Computing model and dynamic graphs

Exercise 1 Give the families of dynamic graphs that "naturally" correspond to:

1. synchronous systems over a complete network and at most f faulty senders;
2. synchronous systems over a complete network and at most f crashes;
3. asynchronous systems over a complete network and at most f crashes;
4. asynchronous systems over a complete network and at most f initial crashes. (Do not forget links are reliable.)

To what of these families of dynamic graphs does the Santoro and Widmayer's theorem apply?

Exercise 2 Show that the product of $|V| - 1$ strongly connected digraphs with the same set of nodes V and a self-loop at each node is the complete digraph.

Exercise 3 Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two digraphs with the same set of nodes V, and let $G = G_1 \circ G_2$. Assume that each node in V has at least $n - f_1$ and $n - f_2$ incoming neighbors in G_1 and G_2, respectively.

1. Show that any node that is not a central root of G has at most f_2 outgoing neighbors in G_1.
2. Use the avove bound to prove that G has at least $n - f_1 \left(1 + \frac{f_2}{n - f_2}\right)$ central roots.
3. Suppose that a dynamic graph $\mathbb{G}$ is such that
$$\forall i \in V, \ \forall t \in \mathbb{N}, \ |\mathrm{In}_i(t)| > |V|/2.$$
With what delay is the dynamic graph $\mathbb{G}$ centered?

Exercise 4 Show that if a dynamic graph $\mathbb{G}$ on the set of nodes V is continuously rooted, then $\mathbb{G}$ is non-split with delay $|V| - 1$. Compare this result with Exercise 2.

Exercise 5 For each positive integer n, let $\lambda(n)$ denote the integer defined by
$$2^{\lambda(n)-1} < n \leqslant 2^{\lambda(n)}.$$
Let V be a finite and non-empty set of cardinality n, and let $G_1 = (V, E_1), \cdots, G_{\lambda(n)} = (V, E_{\lambda(n)})$ be $\lambda(n)$ digraphs with the same set of nodes V and a self-loop at each node. For every index $t \in \{1, \ldots, \lambda(n)\}$, we let
$$G_t^* = G_{\lambda(n)-t+1} \circ \cdots \circ G_{\lambda(n)}.$$

1. What are the digraphs G_1^* and $G_{\lambda(n)}^*$?
2. Prove that if all the digraphs $G_1, \cdots, G_{\lambda(n)}$ are non-split, then every subset of 2^t nodes share a common incoming neighbor in G_t^*, for all indices $t \in \{1, \ldots, \lambda(n)\}$.
3. What property can you then derive for dynamic graphs that are continuously non-split? For dynamic graphs that are continuously rooted (use Exercise 3)?

Exercise 6 Let $\mathbb{G}$ be any dynamic graph., and let $G_1 = (V, E_1), \cdots, G_{\lambda(n)} = (V, E_{\lambda(n)})$ be $\lambda(n)$ digraphs with the same setss of nodes V and a self-loop at each node. For every index $t \in \{1, \ldots, \lambda(n)\}$, we let

$$G_t^* = G_{\lambda(n)-t+1} \circ \cdots \circ G_{\lambda(n)}.$$

1. Show that the digraphs in the integral dynamic graph $\overline{\mathbb{G}}$ eventually stabilize to $\overline{\mathbb{G}}$'s limit superior, i.e.,

$$\exists s,\ \forall t \geqslant s,\ \overline{\mathbb{G}}(t) = \overline{\mathbb{G}}(\infty).$$

2. Prove that $\overline{\mathbb{G}}(\infty)$ is the transitive closure of $\mathbb{G}(\infty)$.

3. Give a caracterization of the kernel of $\mathbb{G}$ in terms of the roots of $\mathbb{G}(\infty)$.

Exercise 7 Use the positive result in the FLP paper to obtain a family of dynamic graphs that can be implemented in any asynchronous system over a complete network and a minority of initial crashes.