Wqos: homework assignment and midterm exam

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Turn your solution in by October 22nd, 2018!

Note: be as clear as possible. If I cannot understand you, this won't help you obtain a good grade. Specific hints: (1) write, do not scribble; (2) if I use a specific notation, use the same; (3) justify every claim you make, by a proof, by a definition, by a previous question, or by a theorem (use theorem *names*, such as "Higman's Lemma", rather than numbers); (4) find the simplest possible proof argument.

Also, the final exam may contain sequels to some of the questions asked here.

1 ω^2-wqos

Recall from the slides of the lectures that an ω^2-*wqo* is a quasi-order $(D, \leq)$ such that every ω^2-sequence $(x_{m,n})_{m<n\in\mathbb{N}}$ on D is ω^2-*good*, in the sense that there are $m < n < p$ in $\mathbb{N}$ such that $x_{m,n} \leq x_{n,p}$. The ω^2-sequences that are not ω^2-good are called ω^2-*bad*.

We order $\mathbb{P}(D)$ by the Hoare quasi-ordering $\leq^\flat$, defined by $A \leq^\flat B$ if and only if for every $a \in A$, there is a $b \in B$ such that $a \leq b$.

1. Assume that $(\mathbb{P}(D), \leq^\flat)$ is not wqo. Given a bad sequence $(A_n)_{n\in\mathbb{N}}$ in D, for each $m \in \mathbb{N}$, find points $x_{m,n}$ in $\downarrow A_m$ and not in $\downarrow A_n$ for every $n > m$. Conclude that $(D, \leq)$ is not ω^2-wqo. This proves the following implication:

$$(D, \leq)\ \omega^2\text{-wqo} \Rightarrow (\mathbb{P}(D), \leq^\flat)\ \text{wqo}.$$

2. Prove the converse implication.

3. Show that $\mathbb{P}(\mathbb{N})$ is wqo under $\leq^\flat$. (What are the downwards-closed subsets of $\mathbb{N}$?). Conclude that $\mathbb{N}$, with its usual ordering, is ω^2-wqo.

4. Why is every ω^2-wqo also wqo?

5. The *infinite Ramsey theorem* states that, given any infinite set X, any finite set C of so-called colors, then for every natural number $k \geq 1$, if you color each subset of X of cardinality k by a color from C, then there is an infinite subset I of X such that every subset of I of cardinality k has the same color. (See the Wikipedia page for a proof.)

Using that, show that, if $(D, \leq)$ is ω^2-wqo, then every ω^2-sequence $(x_{m,n})_{m<n\in\mathbb{N}}$ in D is ω^2-*perfect*, in the sense that there is an infinite increasing sequence $m_0 < m_1 < \cdots < m_k < \cdots$ of natural numbers such that, for all $i < j < k$ in $\mathbb{N}$, $x_{m_i,m_j} \leq x_{m_j,m_k}$.

6. Show the ω^2-*Dickson Lemma*: every finite product of ω^2-wqos is ω^2-wqo. (The product of an empty family is just the one-element set $\{*\}$.)

7. Recall that an ideal in $(D, \leq)$ is a directed, downwards-closed subset of D. The ideals form a poset $(Idl(D), \subseteq)$. Given two finite families $\{I_1, \cdots, I_m\}$ and $\{I'_1, \cdots, I'_n\}$ of ideals, show that $\bigcup_{i=1}^m I_i \subseteq \bigcup_{j=1}^n I'_j$ if and only if $\{I_1, \cdots, I_m\} \subseteq^\flat \{I'_1, \cdots, I'_n\}$. ($\subseteq^\flat$ is the Hoare ordering on $\mathbb{P}_{\text{fin}}(Idl(D))$.)

8. Show that $(D, \leq)$ is ω^2-wqo if and only if its ideal completion $(Idl(D), \subseteq)$ is wqo. Recall that $Idl(D)$ consists of the ideals of D, namely the directed, downwards-closed subsets of D.

2 Embedding on infinite words

An *infinite word* w over D is a map from $\mathbb{N}$ to D. We write w_n for $w(n)$. We also write ww' for the concatenation of the finite word w with the finite or infinite word w'. Let D^ω be the set of infinite words over D.

Let $\leq$ be a qo over D. The *embedding qo* over D^ω is given by $w \leq_\omega w'$ if and only if there is an injective, increasing map $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $w_n \leq w'_{f(n)}$. It is practical to say that f *witnesses* $w \leq_\omega w'$ in that case.

For every $w \in D^\omega$, we write $\overline{w}_n$ for the downwards-closed set $\downarrow\{w_m \mid m \geq n\}$. We write $\limsup w$ for the downwards-closed set $\bigcap_{n\in\mathbb{N}} \overline{w}_n$.

9. Show that, if $(D, \leq)$ is wqo, then for every $w \in D^\omega$, there is a natural number n such that $\limsup w = \overline{w}_n$. The least such n will be called the *looping index* of w. We will also say that an infinite word is *looping* if and only if its looping index is 0.

10. Let $(D, \leq)$ be wqo. Show that, for every $w \in D^\omega$, for every looping word w' in D^ω, $w \leq_\omega w'$ if and only if $\overline{w}_0 \leq^\flat \limsup w'$.

11. Let $(D, \leq)$ be wqo. Show that, for all $w, w' \in D^\omega$, $w \leq_\omega w'$ if and only if that can be deduced using the following rules (by a finite proof; d, d' denote elements of D; w, w' denote elements of D^ω):

$$\frac{w' \text{ looping} \quad \overline{w}_0 \subseteq \limsup w'}{w \leq_\omega w'}\,(\omega) \qquad \frac{w \leq_\omega w'}{w \leq_\omega dw'}\,(add) \qquad \frac{d \leq d' \quad w \leq_\omega w'}{dw \leq_\omega d'w'}\,(inc)$$

12. Given an ω^2-wqo $(D, \leq)$, why is $D^* \times \mathbb{P}_c(D)$ wqo? D^* is equipped with the qo $\leq_*$, and $\mathbb{P}_c(D)$ is the set of countable non-empty subsets of D, quasi-ordered by $\leq^\flat$. A countable non-empty subset if D is any set of the form $\{d_n \mid n \in \mathbb{N}\}$ where each d_n is in D.

13. For every wqo $(D, \leq)$, for every countable non-empty subset A of D, show that there is an infinite, looping word $\tilde{A}$ on D such that $\limsup \tilde{A} = \downarrow A$.

14. Show that if $(D, \leq)$ is ω^2-wqo, then $(D^\omega, \leq_\omega)$ is wqo.

15. Is the assumption that $(D, \leq)$ is ω^2-wqo necessary in the previous question? In other words, is there a qo $(D, \leq)$ such that $(D, \leq)$ is not ω^2-wqo, but $(D^\omega, \leq_\omega)$ is wqo?

3 A weird WSTS

(This is Exercise 1.16 of the lecture notes.)

Consider a deterministic Turing machine $\mathcal{M}$, and let X be its set of configurations. Let $\rightarrow$ be the (one-step) transition relation of $\mathcal{M}$: $C \rightarrow C'$ if running $\mathcal{M}$ for exactly one step from configuration C, we arrive at configuration C'. For each $C \in X$, let $\ell(C)$ be the length of the unique (complete) computation $C \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow \cdots$ of $\mathcal{M}$ starting with configuration C; $\ell(C) = +\infty$ if $\mathcal{M}$ does not terminate starting from $\mathcal{M}$. Let $C \leq C'$ iff $\ell(C) \leq \ell(C')$ in $\mathbb{N}_\omega = \mathbb{N} \cup \{+\infty\}$.

16. Show that $(\mathcal{M}, \rightarrow, \leq)$ is a WSTS.

17. Given a universal Turing machine $\mathcal{M}$, why is the coverability problem of $(\mathcal{M}, \rightarrow, \leq)$ undecidable? The notation I use for the initial configuration of $\mathcal{M}$ on input x is $C(x)$.

18. Show that every universal Turing machine $\mathcal{M}$ can be modified so as to yield another universal Turing machine $\mathcal{M}'$ such that $(\mathcal{M}', \rightarrow, \leq)$ has an effective pred-basis.

19. In the lectures, we have seen that every WSTS satisfying certain conditions has a decidable coverability problem. List those conditions, and say which ones fail in WSTS of the form $(\mathcal{M}', \rightarrow, \leq)$, where $\mathcal{M}'$ is as in the previous question. Justify.