

# Models and Preliminaries

## 1 Synchronous rounds

We introduce:

1. the notion of *rounds*;
2. the notion of *synchronous rounds* with no message loss.

That naturally leads to the *communication graph at round  $t$* , denoted  $\mathbb{G}(t) = (V, \mathbb{E}(t))$ , which is a directed graph with a set of nodes  $V$ , and to the *dynamic graph*  $\mathbb{G} = (\mathbb{G}(t))_{t \in \mathbb{N}^*}$ , which is an infinite sequence of directed graphs over  $V$ .

Then we discussed various system models leading to different ways for producing communication graphs, and so dynamic graphs (external oracles *vs.* endogenous rules).

## 2 Networks, Algorithms, and Specifications

In the following, we fix a non-empty set  $\mathcal{V}$ . For any non-empty set  $V$ , a *valuation of  $V$*  is a mapping  $\mu : V \rightarrow \mathcal{V}$ .

### Networks and classes of networks

A *network* means a valued dynamic graph, i.e., a dynamic graph  $\mathbb{G} = (V, \mathbb{E})$  with a valuation  $\mu$  of  $V$ ; the network is then denoted  $\mathbb{G}^\mu = (V, \mu, \mathbb{E})$ . The nodes of  $\mathbb{G}$  are called *agents*.

Classes of networks specify knowledge: the larger the class is, the smaller the knowledge is. For instance, the class of all networks corresponds to no knowledge while a class reduced to a singleton corresponds to maximum knowledge. Common situations studied in the literature include the knowledge of the whole network (the latter example), of the number of agents (the class of all networks with the same number of agents), of the set of valuations (identifiers), or of some graph-theoretical property.

A class of networks  $\mathcal{C}$  is naturally structured into an infinite tree: each network corresponds to an infinite branch from the root, and every edge in the tree is labelled by the corresponding directed graph. However, it is important to note that an infinite branch from the root may correspond to no network in  $\mathcal{C}$ . The class of networks  $\mathcal{C}$  is said to be *closed* if there is a one-to-one correspondance between the infinite branches from the root and the networks in  $\mathcal{C}$ .

### Algorithms

An *algorithm*  $A$  (for  $\mathcal{V}$ ) is a quintuple  $A = (X, X^0, \mathcal{M}, S, T)$  where

1.  $X$  is a non-empty set;

2.  $X^0$  is a non-empty subset of  $X$ ;
3.  $\mathcal{M}$  is a non-empty set;
4.  $S$  is a mapping  $S : \mathcal{V} \times X \rightarrow \mathcal{M}$ ;
5.  $T$  is a mapping  $T : \mathcal{V} \times X \times \mathcal{M}^\oplus \rightarrow X$ , where  $\mathcal{M}^\oplus$  denotes the set of multisets over  $\mathcal{M}$ .

The elements of  $\mathcal{M}$  are called *messages*. The functions  $S$  and  $T$  are called the *sending function* and the *transition function* of the algorithm, respectively.

A *configuration of  $A$*  for a set of agents  $V$  is any function  $C : V \rightarrow X$ ;  $C$  is said to be an *initial configuration of  $A$*  if  $C : V \rightarrow X^0$ .

An *execution of  $A$  for the network  $\mathbb{G}^\mu$* , where  $\mu$  is a valuation of  $V$ , is an infinite sequence of  $A$ 's configurations for  $V$ ,  $(C(t))_{t \in \mathbb{N}}$ , such that

1.  $C(0)$  is an initial configuration of  $A$ ;
2.  $\forall t \in \mathbb{N}, \forall i \in V, C_i(t+1) = T(\mu(i), C_i(t), M_i(t+1))$ , where  $M_i(t+1)$  is the multiset of messages received by  $i$  in round  $t+1$ , i.e.,  $M_i(t+1) = \langle S(\mu(j), C_j(t)) : j \in \text{In}_i(t+1) \rangle$ .

Observe that  $A$  is deterministic: from any initial configuration  $C_0$ , the algorithm  $A$  has a unique execution for a given network  $\mathbb{G}^\mu$ . The algorithm  $A$  is said to be *self-stabilizing* if it may start from any configuration, that is to say when  $X^0 = X$ .

Let us now fix a set  $V$  of agents, a class of networks  $\mathcal{C}$  over  $V$ , and a valuation of  $V$ . Let  $C$  be any configuration of  $A$  for  $V$ . A configuration  $C'$  is said to be *reachable from  $C$  in  $\mathcal{C}$*  if there exists an execution  $(C(t))_{t \in \mathbb{N}}$  of  $A$  for a network in  $\mathcal{C}$  and two indices  $t$  and  $t'$ ,  $t \leq t'$ , such that  $C(t) = C$  and  $C(t') = C'$ . The nodes in the above mentioned tree associated to  $\mathcal{C}$  may be labelled by the configurations reachable in  $\mathcal{C}$  from an initial configuration  $C_0$ , and every execution of  $A$  from  $C_0$  corresponds to an infinite branch starting at  $C_0$ .

## Specifications

Let  $Y$  be any non-empty set. A *specification on  $Y$*  is a mapping

$$S : \mu \rightarrow S_\mu \subseteq (Y^V)^\omega$$

that assigns a set  $S_\mu$  of infinite sequences of elements in  $Y^V$  to any valuation  $\mu$  of  $V$ . The specification  $S$  is said to be *stabilizing* if each set  $S_\mu$  is suffix-closed.

**Examples.** For every consensus problem, we let

$$Y = \mathcal{V} \times (\mathcal{V} \cup \{\perp\})$$

and we only consider infinite sequences  $y$  such that

$$\forall i \in V, \forall t \in \mathbb{N}, y_i[1](t) = y_i[1](0).$$

Given  $y \in (Y^V)^\omega$ , we let  $\mu_i := y_i[1](0)$  and write  $y_i(t)$  instead of  $y_i[2](t)$  as no confusion may arise.

**Irrevocable consensus.** The set  $S_V$  is the set of infinite sequences such that

Init:  $\forall i \in V, \forall t \in \mathbb{N}, y_i(0) = \perp$ .

Validity:  $\forall i \in V, \forall t \in \mathbb{N}, y_i(t) \neq \perp \Rightarrow y_i(t) \in \{\mu_j : j \in V\}$ .

Termination:  $\forall i \in V, \exists t_i \in \mathbb{N}^*, \forall t \geq t_i, y_i(t) \neq \perp$ .

Irrevocability and Agreement:  $\forall i, j \in V, \forall s, t \in \mathbb{N}, y_i(s) \neq \perp \wedge y_i(t) \neq \perp \Rightarrow y_i(s) = y_i(t)$ .

**Stabilizing consensus.** The set  $S_V$  is the set of infinite sequences such that

Convergence:  $\forall i \in V, \lim_{t \rightarrow \infty} y_i(t)$  exists for the discrete topology.

Validity:  $\forall i \in V, \lim_{t \rightarrow \infty} y_i(t)$  exists  $\Rightarrow \lim_{t \rightarrow \infty} y_i(t) \in \{\mu_j : j \in V\}$ .

Agreement:  $\forall i, j \in V, \lim_{t \rightarrow \infty} y_i(t)$  and  $\lim_{t \rightarrow \infty} y_j(t)$  exist  $\Rightarrow \lim_{t \rightarrow \infty} y_i(t) = \lim_{t \rightarrow \infty} y_j(t)$ .

**Asymptotic consensus.** The value domain is  $\mathcal{V} = \mathbb{R}^d$ . The set  $S_V$  is the set of infinite sequences such that

Convergence:  $\forall i \in V, \lim_{t \rightarrow \infty} y_i(t)$  exists for the euclidean metrics.

Validity:  $\forall i \in V, \lim_{t \rightarrow \infty} y_i(t)$  exists  $\Rightarrow \lim_{t \rightarrow \infty} y_i(t) \in \text{conv}(\mu_j : j \in V)$ .

Agreement:  $\forall i, j \in V, \lim_{t \rightarrow \infty} y_i(t)$  and  $\lim_{t \rightarrow \infty} y_j(t)$  exist  $\Rightarrow \lim_{t \rightarrow \infty} y_i(t) = \lim_{t \rightarrow \infty} y_j(t)$ .

### Solvability in a class of networks

Let  $S$  be a specification on  $Y$ , and let  $\mathcal{C}$  be a class of networks. An algorithm  $A$  is said to *solve*  $S$  in  $\mathcal{C}$  if the state space of  $A$  is of the form  $X \times Y$  and if for every network  $\mathbb{G}^\mu$  in  $\mathcal{C}$  and every execution of  $A$  for  $\mathbb{G}^\mu$

$$(x(0), y(0)), (x(1), y(1)), (x(2), y(2)), \dots$$

the infinite sequence

$$y(0), y(1), y(2), \dots$$

is in  $S_\mu$ .