Asymptotic Consensus and Averaging Algorithms

Exercise 1. Let $\mathcal{S}_n$ be the set of stochastic matrices of size n, and let $\mathcal{M} \subseteq \mathcal{S}_n$ be a non-empty and finite subset of $\mathcal{S}_n$ such that any finite product of matrices in $\mathcal{M}$ is ergodic. We define the equivalence relation $\sim$ in $\mathcal{S}_n$ by:

$$A \sim B \Leftrightarrow G(A) = G(B),$$

where $G(A)$ and $G(B)$ are the graphs associated to A and B, respectively.

1. Show that the relation $\sim$ is preserved by right (or left) multiplication, i.e.,

$$\forall A, B, M \in \mathcal{S}_n, \quad A \sim B \Rightarrow AM \sim BM.$$

2. Suppose that A and B are two equivalent matrices, i.e., $A \sim B$. Prove that $N(A) = 1$ if and only if $N(B) = 1$.

Let $A_0, A_1, \cdots, A_{n^2}$ a sequence of $n^2 + 1$ matrices in $\mathcal{M}$.

3. Show that there exist two indices k and ℓ, $0 \leqslant k < \ell \leqslant n^2$, such that $A_{n^2} \cdots A_k \sim A_{n^2} \cdots A_\ell$.
4. Prove that $A_{n^2} \cdots A_0$ is a scrambling matrix, i.e., $N(A_{n^2} \cdots A_0) < 1$.
5. What extension of Corollary 6 (cf. the course notes) have you just proved?

Exercise 2. A stochastic matrix A is said to be *doubly stochastic* if its transpose A^T is also stochastic.

1. Define a class of stochastic matrices that are all doubly stochastic.
2. What is the Perron vector of a doubly stochastic matrix?

Exercise 3. Let $G = ([n], E)$ be a symmetric and connected graph, and let A be the stochastic matrix such that

$$A_{i,j} = 1/d_i,$$

where $d_i = d_i^- = d_i^+$ is the in-degree (or outdegree) of the node i in G.

1. Show that the i-th entry of the Perron vector of A is equal to $\pi_i = d_i/|E|$.

Let $q_1, \cdots, q_n$ be n integers such that $q_i \geqslant d_i$, and let B the $n \times n$ matrix defined by

$$A_{i,j} = 1/q_i.$$

2. Verify that B is a stochastic matrice.
3. What is the Perron vector of B? What property do the FixedWeight and the Metropolis algorithms share?

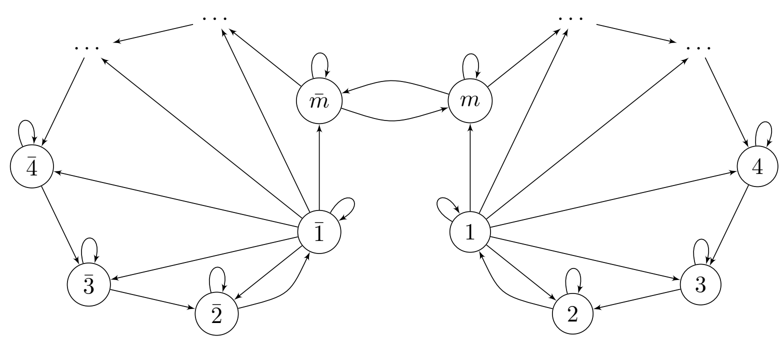

Figure 1: The m-butterfly graph

Exercise 4. Let us consider the *m-butterfly graph* depicted in Figure 1: It has $n = 2m$ nodes and consists of two isomorphic parts that are connected by a bidirectional edge. We list the edges between the nodes $1, 2, \ldots, m$, which also determine the edges between the nodes $m+1, m+2, \ldots, 2m$ via the isomorphism $\bar{p} = 2m - p + 1$. The edges between the nodes $1, 2, \ldots, m$ are: (a) the edges $(p+1, p)$ for all $p \in [m-1]$ and (b) the edges $(1, p)$ for all $p \in [m]$. In addition, it contains a self-loop at each node and the two edges $(m, \bar{m})$ and $(\bar{m}, m)$. Hence the m-butterfly graph is strongly connected.

Let A be the stochastic matrix such that

$$A_{i,j} = 1/d_i^+,$$

where d_i^- is the in-degree of the node i in the Butterfly graph.

1. Verify that A is an ergodic matrix and its Perron vector is given by

$$\pi_1 = \frac{1}{5} \quad , \quad \pi_p = \frac{3}{5 \cdot 2^p} \text{ for } p \in \{2, \ldots, m-1\} \quad \text{and} \quad \pi_m = \frac{3}{5 \cdot 2^{m-1}} \, .$$

2. Compare this Perron Vector with the one in Exercise 3, question 1.

Exercise 5. An averaging algorithm is said to be *α-safe* for a dynamic network $\mathbb{G}$ if, in every execution of this algorithm with the communication network $\mathbb{G}$, all positive weights are at least equal to α.

1. We consider an α-safe averaging algorithm in a dynamic network $\mathbb{G}$, and an execution of this algorithm with $\mathbb{G}$. Prove that at every round t and for every agent i, the output variable x_i satisfies

$$(1-\alpha)m_i(t-1) + \alpha M_i(t-1) \leqslant x_i(t) \leqslant (1-\alpha)M_i(t-1) + \alpha m_i(t-1),$$

where $m_i(t-1) = \min_{j \in In_i(t)} x_j(t-1)$, $M_i(t-1) = \max_{j \in In_i(t)} x_j(t-1)$, and $In_i(t)$ denotes the set of i's incoming neighbors in $\mathbb{G}(t)$.

2. Does the following inequalities:

$$(1-\alpha)m(t-1) + \alpha M(t-1) \leqslant x_i(t) \leqslant (1-\alpha)M(t-1) + \alpha m(t-1),$$

where $m(t-1) = \min_{j \in [n]} x_j(t-1)$, $M(t-1) = \max_{j \in [n]} x_j(t-1)$, hold?

Let $G = ([n], E)$ be a symmetric and connected graph.

3. Is the *EqualNeighbor* algorithm α-safe in $\mathbb{G}$? For what real number α?

4. Same questions for the *FixedWeight* algorithm and the *Metropolis* algorithm.