Course - (GPAC) Generable Functions

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Abstract

The following document is based on [BGP17], in turn based on [Pou15].

Contents

1 Notations

In this document, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ = [0, +\infty)$ the nonnegative real numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$ the natural numbers, $\mathbb{Z}$ the integers, $[\![a, b]\!] = \{a, a+1, \dots, b\}$ the integers between a and b, $\mathbb{Q}$ the rational numbers, $\mathbb{R}_P$ the polynomial time computable real numbers [Ko91], $\mathbb{R}_G$ the smallest generable field (see Section 4). $M_{n,d}(\mathbb{K})$ denotes the set of $n \times d$ matrices over the ring $\mathbb{K}$. For any set X, $\mathcal{P}(X)$ denotes the powerset of X and $\#X$ the cardinal of X. For any function f, $\operatorname{dom} f$ is the domain of f, $f^{[n]}$ the n^{th} iterate of f, $f{\restriction}_X$ the restriction of f to X, $J_f(x)$ denotes the Jacobian matrix of f at x. For any vector $y \in \mathbb{R}^n$ and $e \leqslant n$, $y_{1..e} = (y_1, \dots, y_e)$ denotes the first e components of y and

$\|y\| = \max(|y_1|, \dots, |y_n|)$ denotes the infinity norm. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0) = \{x : \|x - x_0\|_2 < r\}$ denotes the open of radius r and center p for the euclidean norm. Given a (multivariate) polynomial p, $\deg(p)$ denotes its degree and Σp the sum of the absolute value of its coefficients. We denote by $\mathbb{K}[\mathbb{R}^d]$ the set of polynomial functions in d variables with coefficients in $\mathbb{K}$. Given a vector of polynomial $p = (p_1, \dots, p_k)$, which we simply refer to as a polynomial, $\deg(p) = \max(\deg(p_1), \dots, \deg(p_k))$ and $\Sigma p = \max(\Sigma p_1, \dots, \Sigma p_k)$. We denote by $\mathbb{K}^k[\mathbb{R}^d]$ the set of vectors of polynomial functions in d variables of size k with coefficients in $\mathbb{K}$. In this article, we write poly to denote an unspecified polynomial. For any $x \in \mathbb{R}$, $\operatorname{sgn}(x)$ denotes the sign of x, $\lfloor x \rfloor$ the integer part of x, $\operatorname{int}_k(x) = \max(0, \min(k, \lfloor x \rfloor))$, $\lfloor x \rceil$ the nearest integer (undefined for $n + \frac{1}{2}$).

2 Generable functions

In this section, we will define a notion of function generated by a PIVP. From previous discussions, they correspond to functions generated by the General Purpose Analog Computers of Claude Shannon [Sha41];

This class of functions is closed by a number of natural operations such as arithmetic operators or composition. In particular, we will see that those functions are always analytic. The major property of this class is the stability by ODE solving: if f is *generable* and y satisfies $y' = f(y)$ then y is generable. This means that we can design differential systems where the right-hand side contains much more general functions than polynomials, and this system can be rewritten to use polynomials only.

Several of the results here are extensions to the multidimensional case of results established in [Gra07]. Moreover, a noticeable difference is that here we are also talking about complexity, whereas [Gra07] is often not precise about the growth of functions as only motivated by computability theory.

In this section, $\mathbb{K}$ will always refer to a real field, for example $\mathbb{K} = \mathbb{Q}$. The basic definitions work for any such field but the main results will require some assumptions on $\mathbb{K}$. These assumptions will be formalized in Definition 2 and detailed in Section 4.

2.1 Unidimensional case

We start with the definition of generable functions from $\mathbb{R}$ to $\mathbb{R}^n$. Those are defined as the solution of some polynomial IVP (PIVP) with an additional boundedness constraint. This will be of course key to talk about complexity theory for the GPAC, since if no constraint is put on the growth of functions, it is easy to see that arbitrary growing functions can be generated by a GPAC (or, equivalently, by a PIVP), such as the $t \mapsto \exp(\exp(\dots \exp(t)))$ function. Indeed consider the following system

$$\begin{cases} y_1(0)= 1 \\ y_2(0)= 1 \\ \quad\dots \\ y_n(0)= 1 \end{cases} \qquad \begin{cases} y_1'(t)= y_1(t) \\ y_2(t)= y_1(t)y_2(t) \\ \quad\dots \\ y_d'(t)= y_1(t)\cdots y_n(t) \end{cases}$$

This system has the form (**??**) and can be solved explicitly. It has the following solution:

$$y_1(t) = e^t \qquad y_{n+1}(t) = e^{y_n(t)-1} \qquad y_d(t) = e^{e^{\cdot^{\cdot^{\cdot^{e^{e^t}-1}}}}-1}$$

Hence, although previous papers about the GPAC studied computability, like [Sha41], [PE74], [GC03] or [Gra04], they said nothing about complexity. And as the previous example shows, the output of a GPAC can have an arbitrarily high growth and thus arbitrarily high complexity. Hence, to distinguish between reasonable GPACs, it is natural to bound the growth of the outputs of a GPAC and use those bounds as a complexity measure. Moreover, as we have shown in [**?**], we can compute (in the Computable Analysis setting [BHW08]) the solution of a PIVP in time polynomial in the growth bound of the PIVP. This motivates the following definition (in what follows, $\mathbb{K}[\mathbb{R}^n]$ denotes polynomial functions with n variables and with coefficients in $\mathbb{K}$, where variables live in $\mathbb{R}^n$ and[1] $\mathbb{R}_+ = [0, +\infty[$):

Definition 1 (Generable function) *Let* $\mathtt{sp} : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing function and* $f : \mathbb{R} \to \mathbb{R}^m$*. We say that* $f \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$ *if and only if there exists* $n \geqslant m$*,* $y_0 \in \mathbb{K}^n$ *and* $p \in \mathbb{K}^n[\mathbb{R}^n]$ *such that there is a (unique)* $y : \mathbb{R} \to \mathbb{R}^n$ *satisfying for all time* $t \in \mathbb{R}$*:*

- $y'(t) = p(y(t))$ *and* $y(0) = y_0$ ► *y satisfies a differential equation*
- $f(t) = y_{1..m}(t) = (y_1(t), \dots, y_m(t))$ ► *f is a component of y*
- $\|y(t)\| \leqslant \mathtt{sp}(|t|)$ ► *y is bounded by* sp

The set of all generable functions is denoted by $\mathrm{GVAL}_{\mathbb{K}} = \bigcup_{\mathtt{sp}:\mathbb{R}\to\mathbb{R}_+} \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$*. When this is not ambiguous, we do not specify the field* $\mathbb{K}$ *and write* $\mathrm{GVAL}[\mathtt{sp}]$ *or simply* GVAL*. We will also write* GVAL[poly] *(or* $\mathrm{GVAL}_{\mathbb{K}}[\mathrm{poly}]$*) as a synonym of* $\mathrm{GVAL}[\mathtt{sp}]$ *(respectively:* $\mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$*) for some polynomial* sp *(see coming Remark 9).*

Comment 1 (Uniqueness) *The uniqueness of y in Definition 1 is a consequence of the Cauchy-Lipschitz theorem. Indeed a polynomial is a locally Lipschitz function.*

[1] We write $[a, b]$ (respectively: $]a, b]$, $[a, b[$, $]a, b[$) for closed (resp. semi-closed, open) interval.

Comment 2 (Regularity) *As a consequence of the Cauchy-Lipschitz theorem, the solution y in Definition 1 is at least C^∞. It can be seen that it is in fact real analytic, as it is the case for analytic differential equations in general [Arn78].*

Comment 3 (Multidimensional output) *It should be noted that although Definition 1 defines generable functions with output in $\mathbb{R}^m$, it is completely equivalent to say that f is generable if and only if each of its component is (*i.e. *f_i is generable for every i); and restrict the previous definition to functions from $\mathbb{R}$ to $\mathbb{R}$ only. Also note that if y is the solution from Definition 1, then obviously y is generable.*

Although this might not be obvious at first glance, this class contains polynomials, and contains many elementary functions such as the exponential function, as well as the trigonometric functions. Intuitively, all functions in this class can be computed efficiently by classical machines, where $\mathtt{sp}$ measures some "hardness" in computing the function. We took care to choose the constants such as the initial time and value, and the coefficients of the polynomial in $\mathbb{K}$. The idea is to prevent any uncomputability from arising by the choice of uncomputable real numbers in the constants.

Example 1 (Polynomials are generable) *Let p in $\mathbb{Q}(\pi)[\mathbb{R}]$. For example $p(x) = x^7 - 14x^3 + \pi^2$. We will show that $p \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$ where $\mathtt{sp}(x) = x^7 + 14x^3 + \pi^2$. We need to rewrite p with a polynomial differential equation: we immediately get that $p(0) = \pi^2$ and $p'(x) = 7x^6 - 42x^2$. However, we cannot express $p'(x)$ as a polynomial of $p(x)$ only: we need access to x. This can be done by introducing a new variable $v(x)$ such that $v(x) = x$. Indeed, $v'(x) = 1$ and $v(0) = 0$. Finally we get:*

$$\begin{cases} p(0) = \pi^2 \\ p'(x) = 7v(x)^6 - 42v(x)^2 \end{cases} \qquad \begin{cases} v(0) = 0 \\ v'(x) = 1 \end{cases}$$

Formally, we define $y(x) = (p(x), x)$ and show that $y(0) = (\pi^2, 0) \in \mathbb{K}^2$ and $y'(x) = p(y(x))$ where $p_1(a, b) = 7b^6 - 42b^2$ and $p_2(a, b) = 1$. Also note that the coefficients are clearly in $\mathbb{Q}(\pi)$). We also need to check that $\mathtt{sp}$ is a bound on $\|y(x)\|$ (for $x \geq 0$):

$$\|y(x)\| = \max(|x|, |x^7 - 14x^3 + \pi^2|) \leq \mathtt{sp}(x)$$

This shows that $p \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$ and can be generalized to show that any polynomial in one variable is generable.

Example 2 (Some generable elementary functions) *We will check that* $\exp \in \mathrm{GVAL}_{\mathbb{Q}}[\exp]$ *and* $\sin, \cos, \tanh \in \mathrm{GVAL}_{\mathbb{Q}}[x \mapsto 1]$. *We will also check that* $\arctan \in \mathrm{GVAL}_{\mathbb{Q}}[x \mapsto \max(x, \frac{\pi}{2})]$.

- *A characterization of the exponential function is the following:* $\exp(0) = 1$ *and* $\exp' = \exp$. *Since* $\|\exp\| = \exp$, *it is immediate that* $\exp \in \mathrm{GVAL}_{\mathbb{Q}}[\exp]$. *The exponential function might be the simplest generable function.*

- *The sine and cosine functions are related by their derivatives since* $\sin' = \cos$ *and* $\cos' = -\sin$. *Also* $\sin(0) = 0$ *and* $\cos(0) = 1$, *and* $\|(\sin(x), \cos(x))\| \leqslant 1$, *we get that* $\sin, \cos \in \mathrm{GVAL}_{\mathbb{Q}}[x \mapsto 1]$ *with the same system.*

- *The hyperbolic tangent function will be very useful in this paper. Is it known to satisfy the very simple polynomial differential equation* $\tanh' = 1 - \tanh^2$. *Since* $\tanh(0) = 0$ *and* $|\tanh(x)| \leqslant 1$, *this shows that* $\tanh \in \mathrm{GVAL}_{\mathbb{Q}}[x \mapsto 1]$.

- *Another very useful function will be the arctangent function. A possible definition of the arctangent is the unique function satisfying* $\arctan(0) = 0$ *and* $\arctan'(x) = \frac{1}{1+x^2}$. *Unfortunately this is neither a polynomial in* $\arctan(x)$ *nor in* x. *A common trick is to introduce a new variable* $z(x) = \frac{1}{1+x^2}$ *so that* $\arctan'(x) = z(x)$, *in the hope that* z *satisfies a PIVP. This is the case since* $z(0) = 1$ *and* $z'(x) = \frac{-2x}{(1+x^2)^2} = -2xz(x)^2$ *which is a polynomial in* z *and* x. *We introduce a new variable for* x *as we did in the previous examples. Finally, define* $y(x) = (\arctan(x), \frac{1}{1+x^2}, x)$ *and check that* $y(0) = (0, 1, 0)$ *and* $y'(x) = (y_2(x), -2y_3(x)y_2(x)^2, 1)$. *The* $\frac{\pi}{2}$ *bound on* $\arctan$ *is a textbook property, and the bound on the other variables is immediate.*

Not only the class of generable functions contains many classical and useful functions, but it is also closed under many operations. We will see that the sum, difference, product and composition of generable functions are still generable.

2.1.1 The issue of constants

Before moving on to the properties of this class, we need to mention the easily overlooked issue about constants, best illustrated as an example.

Example 3 (The issue of constants) *Let* $\mathbb{K}$ *be a field, containing at least the rational numbers. Assume that generable functions are closed under composition, that is for any two* $f, g \in \mathrm{GVAL}_{\mathbb{K}}$ *we have* $f \circ g \in \mathrm{GVAL}_{\mathbb{K}}$. *Let* $\alpha \in \mathbb{K}$ *and* $g = x \mapsto \alpha$. *Then for any* $(f : \mathbb{R} \to \mathbb{R}) \in \mathrm{GVAL}_{\mathbb{K}}$, $f \circ g \in \mathrm{GVAL}_{\mathbb{K}}$. *Using Definition 1, we get that* $f(g(0)) \in \mathbb{K}$ *which means* $f(\alpha) \in \mathbb{K}$ *for any* $\alpha \in \mathbb{K}$. *In other words,* $\mathbb{K}$ *must satisfy the following property:*

$$f(\mathbb{K}) \subseteq \mathbb{K} \qquad \forall f \in \mathrm{GVAL}_{\mathbb{K}}$$

This property does not hold for general fields.

The example above outlines the need for a stronger hypothesis on $\mathbb{K}$ if we want to be able to compose functions. Motivated by this example, we introduce the following notion of *generable field*.

Definition 2 (Generable field) *A field* $\mathbb{K}$ *is* generable *if and only if* $\mathbb{Q} \subseteq \mathbb{K}$ *and for any* $\alpha \in \mathbb{K}$ *and* $(f : \mathbb{R} \to \mathbb{R}) \in \mathrm{GVAL}_{\mathbb{K}}$, *we have* $f(\alpha) \in \mathbb{K}$.

⚠ From now on, we will assume that $\mathbb{K}$ is a generable field. See Section 4 for more details on this assumption.

Example 4 (Usual constants are generable) *In this paper, we will use again and again that some well-known constants belong to any generable field. We detail the proof for π and e:*

- *It is well-known that $\frac{\pi}{4} = \arctan(1)$. We saw in Example 2 that $\arctan \in \mathrm{GVAL}_{\mathbb{Q}}$ and since $1 \in \mathbb{K}$ we get that $\frac{\pi}{4} \in \mathbb{K}$ because $\mathbb{K}$ is a generable field. We conclude that $\pi \in \mathbb{K}$ because $\mathbb{K}$ is a field and $4 \in \mathbb{K}$.*
- *By definition, $e = \exp(1)$ and $\exp \in \mathrm{GVAL}_{\mathbb{Q}}$, so $e \in \mathbb{K}$ because $\mathbb{K}$ is a generable field and $1 \in \mathbb{K}$.*

2.1.2 Robustness of the class

Lemma 1 (Arithmetic on generable functions) *Let $f \in \mathrm{GVAL}[\mathtt{sp}]$ and $g \in \mathrm{GVAL}[\overline{\mathtt{sp}}]$.*

- $f + g, f - g \in \mathrm{GVAL}[\mathtt{sp} + \overline{\mathtt{sp}}]$
- $fg \in \mathrm{GVAL}[\max(\mathtt{sp}, \overline{\mathtt{sp}}, \mathtt{sp}\,\overline{\mathtt{sp}})]$
- $\frac{1}{f} \in \mathrm{GVAL}[\max(\mathtt{sp}, \mathtt{sp}')]$ *where* $\mathtt{sp}'(t) = \frac{1}{|f(t)|}$, *if f never cancels*
- $f \circ g \in \mathrm{GVAL}[\max(\overline{\mathtt{sp}}, \mathtt{sp} \circ \overline{\mathtt{sp}})]$

Note that the first three items only require that $\mathbb{K}$ is a field, whereas the last item also requires $\mathbb{K}$ to be a generable field.

Proof: Assume that $f : \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R} \to \mathbb{R}^\ell$. We will make a detailed proof of the product and composition cases, since the sum and difference are much simpler. The intuition follows from basic differential calculus and the chain rule: $(fg)' = f'g + fg'$ and $(f \circ g)' = g'(f' \circ g)$. Note that $\ell = 1$ for the composition to make sense and $\ell = m$ for the product to make sense (componentwise). The only difficulty in this proof is technical: the differential equation may include more variables than just the ones computing f and g. This requires a bit of notation to stay formal. Apply Definition 1 to f and g to get $p, \overline{p}, y_0, \overline{y}_0$. Consider the following systems:

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \\ \overline{y}(0) = \overline{y}_0 \\ \overline{y}'(t) = \overline{p}(\overline{y}(t)) \end{cases} \qquad \begin{cases} z_i(0) = y_{0,i}\overline{y}_{0,i} \\ z_i'(t) = p_i(y(t))\overline{y}_i(t) + y_i(t)\overline{p}_i(\overline{y}(t)) \\ u_i(0) = f_i(\overline{y}_{0,1}) \\ u_i'(t) = \overline{p}_i(\overline{y}(t))p(u(t)) \end{cases} \qquad i \in [\![1, m]\!]$$

Those systems are clearly polynomial. By construction, u and z exist over $\mathbb{R}$ since $z_i(t) = y_i(t)\overline{y}_i(t)$ satisfies the differential equation over $\mathbb{R}$ (indeed y and $\overline{y}$ exist over $\mathbb{R}$). Similarly, $u_i(t) = y_i(\overline{y}(t))$ exists over $\mathbb{R}$ and satisfies the equation. Remember that by definition, for any $i \in [\![1, m]\!]$ and $j \in [\![1, \ell]\!]$, $f_i(t) = y_i(t)$ and $g_j(t) = z_j(t)$. Consequently, $z_i(t) = f_i(t)g_i(t)$ and $u_i(t) = f_i(g_1(t))$.

Also by definition, $\|y(t)\| \leqslant \mathtt{sp(t)}$ and $\|\overline{y}(t)\| \leqslant \overline{\mathtt{sp}}(t)$. It follows that $|z_i(t)| \leqslant |y_i(t)||\overline{y}_i(t)| \leqslant \mathtt{sp}(t)\overline{\mathtt{sp}}(t)$, and similarly we have $|u_i(t)| \leqslant |f_i(g_1(t))| \leqslant \mathtt{sp}(g_1(t)) \leqslant \mathtt{sp}(\overline{\mathtt{sp}}(t))$.

The case of $\frac{1}{g}$ is very similar: define $g = \frac{1}{f}$ then $g' = -f'g^2$. The only difference is that we don't have an a priori bound on g except $\frac{1}{|f|}$, and we must assume that f is never zero for g to be defined over $\mathbb{R}$.

Finally, a very important note about constants and coefficients which appear in those systems. It is clear that $y_{0,i}\overline{y}_{0,i} \in \mathbb{K}$ because $\mathbb{K}$ is a field. Similarly, for $\frac{1}{f}$ we have $\frac{1}{f(0)} = \frac{1}{y_{0,1}} \in \mathbb{K}$. However, there is no reason in general for $f_i(\overline{y}_{0,1})$ to belong to $\mathbb{K}$, and this is where we need the assumption that $\mathbb{K}$ is generable.
□

2.2 Multidimensional case

We introduced generable functions as a special kind of function from $\mathbb{R}$ to $\mathbb{R}^n$. We saw that this class nicely contains polynomials, however it comes with two defects which prevents other interesting functions from being generable:

- The domain of definition is $\mathbb{R}$: this is very strong, since other "easy" targets such as tan, log or even $x \mapsto \frac{1}{x}$ cannot be defined, despite satisfying polynomial differential equations.
- The domain of definition is one-dimensional: it would be useful to define generable functions in several variables, like multivariate polynomials.

The first issue can be dealt with by adding restrictions on the domain where the differential equation holds, and by shifting the initial condition (0 might not belong to the domain). Overcoming the second problem is less obvious.

2.2.1 About motivation of definitions

The examples below give two intuitions before introducing the formal definition. The first example draws inspiration from multivariate calculus and differential form theory. The second example focuses on GPAC composition. As we will see, both examples highlight the same properties of multidimensional generable functions.

Example 5 (Multidimensional GPAC) *The history and motivation for the GPAC have been described above. The GPAC is the starting point for the definition of generable functions. It crucially relies on the integrator unit to build interesting circuits. In modern terms, the integration is often done implicitly with respect to time, as shown in Figure 1 where the corresponding equation is* $f(t) = \int f$*, or* $f' = f$*. Notice that the circuit has a single "floating input" which is* t *and is only used in the "derivative port" of the integrator. What would be the*

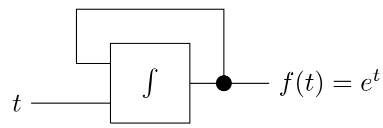

Figure 1: Simple GPAC

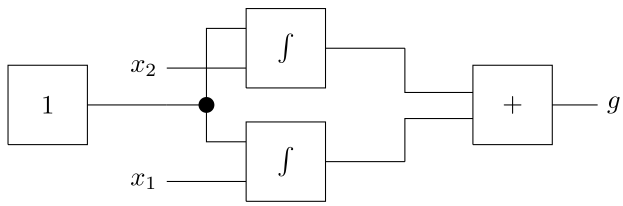

Figure 2: GPAC with two inputs

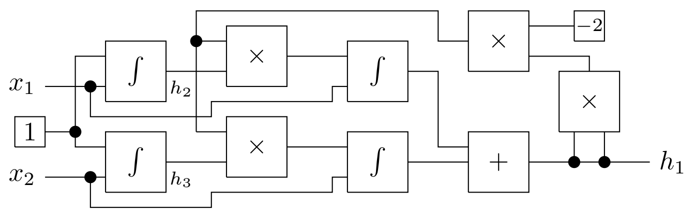

Figure 3: A more involved multidimensional GPAC

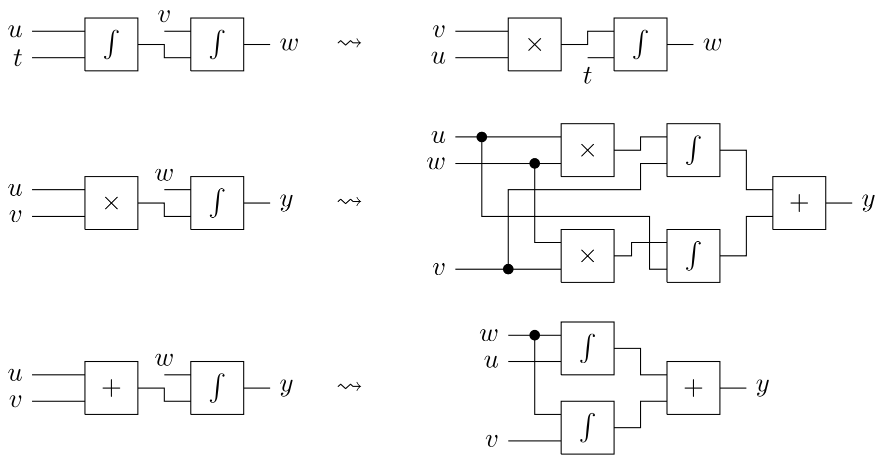

Figure 4: GPAC rewriting

meaning of a circuit with several such inputs, as shown in Figure 2 ? Formally writing the system and differentiating gives:

$$g = \int 1dx_1 + \int 1dx_2 = x_1 + x_2$$
$$dg = dx_1 + dx_2$$

Figure 3 gives a more interesting example to better grasp the features of these GPAC. Using the same "trick" as before we get:

$$\begin{array}{llll} h_2 & = \int 1dx_1 & dh_2 & = dx_1 \\ h_3 & = \int 1dx_2 & dh_3 & = dx_2 \\ h_1 & = \int -2h_1^2h_2dx_1 + \int -2h_1^2h_3dx_2 & dh_1 & = -2h_1^2h_2dx_1 - 2h_1^2h_3dx_2 \end{array}$$

It is now apparent that the computed function h satisfies a special property because $dh_1(x) = p_1(h_1, h_2, h_3)dx_1 + p_2(h_1, h_2, h_3)dx_2$ where p_1 and p_2 are polynomials. In other words, $dh_1 = p(h) \cdot dx$ where $h = (h_1, h_2, h_3)$, $x = (x_1, x_2)$ and $p = (p_1, p_2)$ is a polynomial vector. We obtain similar equations for h_2 and h_3. Finally, $dh = q(h)dx$ where $q(h)$ is the polynomial matrix given by:

$$q(h) = \begin{pmatrix} -2h_1^2h_2 & -2h_1^2h_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This can be equivalently stated as $J_h = q(h)$. This is a generalization of PIVP to polynomial partial differential equations.

To complete this example, note that it can be solved exactly and $h_1(x_1, x_2) = \frac{1}{x_1^2+x_2^2}$ which is defined over $\mathbb{R}^2 \setminus \{(0,0)\}$.

Example 6 (GPAC composition) *Another way to look at Figure 3 and Figure 2 is to imagine that $x_1 = X_1(t)$ and $x_2 = X_2(t)$ are functions of the time (produced by other GPACs), and rewrite the system in the time domain with $h = H(t)$:*

$$\begin{array}{ll} H_2'(t) & = X_1'(t) \\ H_3'(t) & = X_2'(t) \\ H_1'(t) & = -2H_1(t)^2H_2(t)X_1'(t) - 2H_1(t)^2H_3(t)X_2'(t) \end{array}$$

We obtain a system similar to the unidimensional PIVP: for a given choice of X we have $H'(t) = q(H(t))X'(t)$ where $q(h)$ is the polynomial matrix given by:

$$q(h) = \begin{pmatrix} -2h_1^2h_2 & -2h_1^2h_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that this is the same polynomial matrix as in the previous example. The relationship between the time domain H and the original h is simply given by $H(t) = h(x(t))$. This approach has a natural interpretation on the GPAC circuit *in terms of circuit rewriting. Assume that x_1 and x_2 are the outputs of two GPACs (with input t), i.e. $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Then x_1, x_2 are given by the first two components of a polynomial ODE (**??**), i.e. $x_1(t) = y_1(t)$ and $x_2(t) = y_2(t)$. Moreover one has $x_1'(t) = p_1(y), x_2'(t) = p_2(y)$. That means that the output $H(t) = (H_1(t), H_2(t), H_3(t))$ of the GPAC of Figure 3 satisfies*

$$H'(t) = q(H(t))X'(t) = q(H(t))(p_1(y), p_2(y))$$

and therefore consists of the first three components of the polynomial ODE given by

$$\begin{aligned} H' &= q(H(t))(p_1(y), p_2(y)) \\ y' &= p(y) \end{aligned}$$

Thus, if x_1 and x_2 are the outputs of the some GPACs, depending on one input t, and if we connect the outputs of these two GPACs to the inputs of the two-dimensional GPAC of Figure 3, we obtain a one-input GPAC computing $H(t)$, where t is the input. Note that in a normal GPAC, the time t is the only valid input of the derivative port of the integrator, so we need to rewrite integrators which violate this rule. This can be done by rewriting the ODE defining $H(t)$ into a polynomial ODE as done above, and then by implementing a GPAC which computes the solution of this ODE such that the time t is the only valid input of the derivative port of each integrator (this is trivial to implement). This procedure always stops in finite time. Moreover it always works as long as $q(\cdot)$ is a matrix consisting of polynomials.

2.2.2 Formal definitions

These considerations lead to state that the following generalization is clearly the one we want:

Definition 3 (Generable function) *Let $d, \ell \in \mathbb{N}$, I an open and connected subset of $\mathbb{R}^d$, $\mathtt{sp} : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing function and $f : I \to \mathbb{R}^\ell$. We say that $f \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$ if and only if there exists $n \geqslant \ell$, $p \in M_{n,d}(\mathbb{K})[\mathbb{R}^n]$, $x_0 \in (\mathbb{K}^d \cap I)$, $y_0 \in \mathbb{K}^n$ and $y : I \to \mathbb{R}^n$ satisfying for all $x \in I$:*

- *$y(x_0) = y_0$ and $J_y(x) = p(y(x))$ (i.e. $\partial_j y_i(x) = p_{ij}(y(x))$) ► y satisfies a differential equation*
- *$f(x) = y_{1..\ell}(x)$ ► f is a component of y*
- *$\|y(x)\| \leqslant \mathtt{sp}(\|x\|)$ ► y is bounded by $\mathtt{sp}$*

Comment 4 (Uniqueness) *The uniqueness of y in Definition 3 can be seen in two different ways: by uniqueness of the unidimensional case and by analyticity. Note that the existence of y (and thus the domain of definition) is a hypothesis of the definition.*

Consider $x \in I$ and γ a smooth curve[2] from x_0 to x with values in I and consider $z(t) = y(\gamma(t))$ for $t \in [0,1]$. It can be seen that $z'(t) = J_y(\gamma(t))\gamma'(t) =$

[2] see Remark 6

$p(y(\gamma(t))\gamma'(t) = p(z(t))\gamma'(t)$, $z(0) = y(x_0) = y_0$ and $z(1) = y(x)$. The initial value problem $z(0) = y_0$ and $z'(t) = p(z(t))\gamma'(t)$ satisfies the hypothesis of the Cauchy-Lipschitz theorem and as such admits a unique solution. Since this IVP is independent of y, the value of $z(1)$ is unique and must be equal to $y(x)$, for any solution y and any x. This implies that y must be unique.

Alternatively, use Proposition **??** *to conclude that any solution must be analytic. Assume that there are two solutions y and z. Then all partial derivatives at any order at the initial point x_0 are equal because they only depend on y_0. Thus y and z have the same partial derivatives at all order and must be equal on a small open ball around y_0. A classical argument of finite covering with open balls then extends this argument to any point of the interior of domain of definition that is connected to y_0. Since the domain of definition is assumed to be open and connected, this concludes to the equality of y and z.*

Comment 5 (Regularity) *In the euclidean space $\mathbb{R}^n$, C^k smoothness is equivalent to the smoothness of the order k partial derivatives. Consequently, the equation $J_y = p(y)$ on the open set I immediately proves that y is C^∞. Proposition* **??** *shows that y is in fact real analytic.*

Comment 6 (Domain of definition) *Definition 3 requires the domain of definition of f to be connected, otherwise it would not make sense. Indeed, we can only define the value of f at point u if there exists a path from x_0 to u in the domain of f. It could seem, at first sight, that the domain being "only" connected may be too weak to work with. This is not the case, because in the euclidean space $\mathbb{R}^d$,* open *connected subsets are always smoothly arc connected, that is any two points can be connected using a smooth C^1 (and even C^∞) arc. Proposition 3 extends this idea to generable arcs, with a very useful corollary.*

Comment 7 (Multidimensional output) *Remark 3 also applies to this definition: $f :\subseteq \mathbb{R}^d \to \mathbb{R}^n$ is generable if and only if each of its component is generable (*i.e.* f_i is generable for all i).*

Comment 8 (Definition consistency) *It should be clear that Definition 3 and Definition 1 are consistent. More precisely, in the case of unidimensional function ($d = 1$) with domain of definition $I = \mathbb{R}$, both definitions are exactly the same since $J_y = y'$ and $M_{n,1}(\mathbb{R}) = \mathbb{R}^n$.*

The following example focuses on the second issue mentioned at the beginning of the section, namely the domain of definition.

Example 7 (Inverse and logarithm functions) *We illustrate that the choice of the domain of definition makes important differences in the nature of the function.*

- *Let $0 < \varepsilon < 1$ and define $f_\varepsilon : x \in]\varepsilon, \infty[\mapsto \frac{1}{x}$. It can be seen that $f_\varepsilon'(x) = -f_\varepsilon(x)^2$ and $f_\varepsilon(1) = 1$. Furthermore, $|f_\varepsilon(x)| \leqslant \frac{1}{\varepsilon}$ thus $f_\varepsilon \in \mathrm{GVAL}[\alpha \mapsto \frac{1}{\varepsilon}]$. So in particular, $f_\varepsilon \in \mathrm{GVAL}[\mathrm{poly}]$ for any $\varepsilon > 0$. Something interesting arises when $\varepsilon \to 0$: define $f_0(x) = x \in (0, \infty) \mapsto \frac{1}{x}$.*

Then f_0 is still generable and $|f_0(x)| \leqslant \frac{1}{|x|}$. Thus $f_0 \in \mathrm{GVAL}[\alpha \mapsto \frac{1}{\alpha}]$ but $f_0 \notin \mathrm{GVAL}[\mathrm{poly}]$. Note that strictly speaking, $f_0 \in \mathrm{GVAL}[\mathtt{sp}]$ where $\mathtt{sp}(\alpha) = \frac{1}{\alpha}$ and $\mathtt{sp}(0) = 0$ because the bound function needs to be defined over $\mathbb{R}_+$.

- *A similar phenomenon occurs with the logarithm: define $g_\varepsilon : x \in (\varepsilon, \infty) \mapsto \ln(x)$. Then $g'_\varepsilon(x) = f_\varepsilon(x)$ and $g_\varepsilon(1) = 0$. Furthermore, $|g_\varepsilon(x)| \leqslant \max(|x|, |\ln\varepsilon|)$. Thus $g_\varepsilon \in \mathrm{GVAL}[\alpha \mapsto \max(\alpha, |\ln\varepsilon|, \frac{1}{\varepsilon})]$, and in particular $g_\varepsilon \in \mathrm{GVAL}[\mathrm{poly}]$ for any $\varepsilon > 0$. Similarly, $g_0 : x \in]0, \infty[\mapsto \ln(x)$ is generable but does not belong to $\mathrm{GVAL}[\mathrm{poly}]$.*

Example 8 (Classical non-generable functions) *While many of the usual real functions are known to be generated by a GPAC, a notable exception is Euler's Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ function or Riemann's Zeta function $\zeta(x) = \sum_{k=0}^\infty \frac{1}{k^x}$ [Sha41], [PER89]. Furthermore, Riemann's Zeta function (over, for example, $[2, \infty)$) is an example of real-analytic, polynomially-bounded that is not in $\mathrm{GVAL}[\mathrm{poly}]$.*

Example 9 (Generable functions not in $\mathrm{GVAL}[\mathrm{poly}]$) *We have seen that Riemann's Zeta function ζ is an example of a function not in $\mathrm{GVAL}[\mathrm{poly}]$ due to the fact that it is not generable. An example of a generable function not belonging to $\mathrm{GVAL}[\mathrm{poly}]$ is the exponential e^x because, while it is generable, its derivative is not bounded by another polynomial. Note that it is quite possible to have bounded generable functions which do not belong to $\mathrm{GVAL}[\mathrm{poly}]$. An example is the function given by $f(x) = \sin(e^x)$ which is generable and bounded, but its derivative $f'(x) = e^x \cos(e^x)$ is not bounded by any polynomial.*

The previous examples show that $\mathrm{GVAL}_\mathbb{K}[\mathtt{sp}]$ can be used to define a proper hierarchy of generable functions. Adapting the examples given in Example 9 one can show for instance that

$$\mathrm{GVAL}[\mathrm{poly}] \subsetneq \mathrm{GVAL}[e^x] \subsetneq \mathrm{GVAL}[e^{e^x}] \subsetneq \dots$$

In particular these examples show the following result.

Theorem 1 (Existence of noncollapsing classes) $\mathrm{GVAL}[\mathrm{poly}] \subsetneq \mathrm{GVAL}$*.*

3 Stability properties

In this section, the major results will the be stability of multidimensional generable functions under arithmetical operators, composition and ODE solving. Note that some of the results use properties on $\mathbb{K}$ which can be found in Section 4.1.

Lemma 2 (Arithmetic on generable functions) *Let $d, \ell, n, m \in \mathbb{N}$, $\mathtt{sp}, \overline{\mathtt{sp}} : \mathbb{R} \to \mathbb{R}_+$, $f :\subseteq \mathbb{R}^d \to \mathbb{R}^n \in \mathrm{GVAL}[\mathtt{sp}]$ and $g :\subseteq \mathbb{R}^\ell \to \mathbb{R}^m \in \mathrm{GVAL}[\overline{\mathtt{sp}}]$. Then:*

- *$f + g, f - g \in \mathrm{GVAL}[\mathtt{sp} + \overline{\mathtt{sp}}]$ over $\operatorname{dom} f \cap \operatorname{dom} g$ if $d = \ell$ and $n = m$*

- $fg \in \mathrm{GVAL}[\max(\mathtt{sp}, \overline{\mathtt{sp}}, \mathtt{sp}\,\overline{\mathtt{sp}})]$ *if* $d = \ell$ *and* $n = m$
- $f \circ g \in \mathrm{GVAL}[\max(\overline{\mathtt{sp}}, \mathtt{sp} \circ \overline{\mathtt{sp}})]$ *if* $m = d$ *and* $g(\operatorname{dom} g) \subseteq \operatorname{dom} f$

Proof: We focus on the case of the composition, the other cases are very similar.

Apply Definition 3 to f and g to respectively get $l, \bar{l} \in \mathbb{N}$, $p \in M_{l,d}(\mathbb{K})[\mathbb{R}^l]$, $\bar{p} \in M_{\bar{l},\ell}(\mathbb{K})[\mathbb{R}^{\bar{l}}]$, $x_0 \in \operatorname{dom} f \cap \mathbb{K}^d$, $\bar{x}_0 \in \operatorname{dom} g \cap \mathbb{K}^\ell$, $y_0 \in \mathbb{K}^l$, $\bar{y}_0 \in \mathbb{K}^{\bar{l}}$, $y : \operatorname{dom} f \to \mathbb{R}^l$ and $\bar{y} : \operatorname{dom} g \to \mathbb{R}^{\bar{l}}$. Define $h = y \circ g$, then $J_h = J_y(g)J_g = p(h)\bar{p}_{1..m}(\bar{y})$ and $h(\bar{x_0}) = y(\bar{y_0}) \in \mathbb{K}^l$ by Corollary 2. In other words $(\bar{y}, h)$ satisfy:

$$\begin{cases} \bar{y}(\bar{x}_0) = y_0 \in \mathbb{K}^{\bar{l}} \\ h(\bar{x}_0) = y(\bar{y}_0) \in \mathbb{K}^l \end{cases} \qquad \begin{cases} \bar{y}' = \bar{p}(\bar{y}) \\ h' = p(h)\bar{p}_{1..m}(\bar{y}) \end{cases}$$

This shows that $f \circ g = z_{1..m} \in \mathrm{GVAL}$. Furthermore,

$$\begin{aligned} \|(\bar{y}(x), h(x))\| &\leqslant \max(\|\bar{y}(x)\|, \|y(g(x))\|) \\ &\leqslant \max(\overline{\mathtt{sp}}(\|x\|), \mathtt{sp}(\|g(x)\|)) \\ &\leqslant \max(\overline{\mathtt{sp}}(\|x\|), \mathtt{sp}(\overline{\mathtt{sp}}(\|x\|))). \end{aligned}$$

□

Our main result is that the solution to an ODE whose right hand-side is generable, and possibly depends on an external and C^1 control, may be rewritten as a GPAC. A corollary of this result is that the solution to a generable ODE is generable.

Proposition 1 (Generable ODE rewriting) *Let* $d, n \in \mathbb{N}$*,* $I \subseteq \mathbb{R}^n$*,* $X \subseteq \mathbb{R}^d$*,* $\mathtt{sp} : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $(f : I \times X \to \mathbb{R}^n) \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$*. Define* $\overline{\mathtt{sp}} = \max(\mathrm{id}, \mathtt{sp})$*. Then there exists* $m \in \mathbb{N}$*,* $(g : I \times X \to \mathbb{R}^m) \in \mathrm{GVAL}_{\mathbb{K}}[\overline{\mathtt{sp}}]$ *and* $p \in \mathbb{K}^m[\mathbb{R}^m \times \mathbb{R}^d]$ *such that for any interval* J*,* $t_0 \in \mathbb{K} \cap J$*,* $y_0 \in \mathbb{K}^n \cap J$*,* $y \in C^1(J, I)$ *and* $x \in C^1(J, X)$*, if* y *satisfies:*

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = f(y(t), x(t)) \end{cases} \qquad \forall t \in J$$

then there exists $z \in C^1(J, \mathbb{R}^m)$ *such that:*

$$\begin{cases} z(t_0) = g(y_0, x(t_0)) \\ z'(t) = p(z(t), x'(t)) \end{cases} \qquad \begin{cases} y(t) = z_{1..d}(t) \\ \|z(t)\| \leqslant \overline{\mathtt{sp}}(\max(\|y(t)\|, \|x(t)\|)) \end{cases} \qquad \forall t \in J$$

Proof: Apply Definition 3 to f get $m \in \mathbb{N}$, $p \in M_{m,n+d}(\mathbb{K})[\mathbb{R}^m]$, $f_0 \in \operatorname{dom} f \cap \mathbb{K}^d$, $w_0 \in \mathbb{K}^m$ and $w : \operatorname{dom} f \to \mathbb{R}^m$ such that $w(f_0) = w_0$, $J_{w(v)} = p(w(v))$, $\|w(v)\| \leqslant \mathtt{sp}(\|v\|)$ and $w_{1..n}(v) = f(v)$ for all $v \in \operatorname{dom} f$. Define $u(t) = w(y(t), x(t))$, then:

$$\begin{aligned} u'(t) &= J_w(y(t), x(t))(y'(t), x'(t)) \\ &= p(w(y(t), x(t)))(f(y(t), x(t)), x'(t)) \\ &= p(u(t))(u_{1..n}(t), x'(t)) \\ &= q(u(t), x'(t)) \end{aligned}$$

where $q \in \mathbb{K}^m[\mathbb{R}^{m+d}]$ and $u(t_0) = w(y(t_0)) = w(y_0, x(t_0))$. Note that w itself is a generable function and more precisely $w \in \mathrm{GVAL}_{\mathbb{K}}[\mathrm{poly}]$ by definition. Finally, note that $y'(t) = u_{1..d}(t)$ so that we get for all $t \in J$:

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = u_{1..d}(t) \end{cases} \qquad \begin{cases} u(t_0) = w(y_0, x(t_0)) \\ u'(t) = q(u(t), x'(t)) \end{cases}$$

Define $z(t) = (y(t), u(t))$, then $z(t_0) = (y_0, w(y_0, x(t_0))) = g(y_0, x(t_0))$ where $y_0 \in \mathbb{K}^n$ and $w \in \mathrm{GVAL}_{\mathbb{K}}[\mathtt{sp}]$ so $g \in \mathrm{GVAL}_{\mathbb{K}}[\overline{\mathtt{sp}}]$. And clearly $z'(t) = r(z(t), x'(t))$ where $r \in \mathbb{K}^{n+m}[\mathbb{R}^{n+m}]$. Finally, $\|z(t)\| = \max(\|y(t)\|, \|w(y(t), x(t))\|) \leqslant \max(\|y(t)\|, \mathtt{sp}(\max(\|y(t)\|, \|x(t)\|)) \leqslant \overline{\mathtt{sp}}(\max(\|y(t)\|, \|x(t)\|))$. □

A simplified version of this lemma shows that generable functions are closed under ODE solving.

Corollary 1 (Generable functions are closed under ODE) *Let $d \in \mathbb{N}$, $J \subseteq \mathbb{R}$ an interval, $\mathtt{sp}, \overline{\mathtt{sp}} : \mathbb{R}_+ \to \mathbb{R}_+$, $f :\subseteq \mathbb{R}^d \to \mathbb{R}^d$ in $\mathrm{GVAL}[\mathtt{sp}]$, $t_0 \in \mathbb{K} \cap J$ and $y_0 \in \mathbb{K}^d \cap \operatorname{dom} f$. Assume there exists $y : J \to \operatorname{dom} f$ satisfying for all $t \in J$:*

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = f(y(t)) \end{cases} \qquad \|y(t)\| \leqslant \overline{\mathtt{sp}}(t)$$

Then $y \in \mathrm{GVAL}[\max(\overline{\mathtt{sp}}, \mathtt{sp} \circ \overline{\mathtt{sp}})]$ and is unique.

Comment 9 (Polynomially bounded generable functions) *In light of the stability properties above, the class of* polynomially bounded *generable functions,*

$$\mathrm{GVAL}[\mathrm{poly}] = \bigcup_{k=1}^{\infty} \mathrm{GVAL}[\alpha \mapsto k\alpha^k]$$

is particularly interesting because it is stable by operations: addition, multiplication, composition and ODE solving (provided the solution is polynomially bounded). Notice that $\mathrm{GVAL}[\mathrm{poly}]$ is not simply the intersection of GVAL with the set of functions bounded by a polynomial, as shown in Example 9.

Our last result is simple but very useful. Generable functions are continuous and continuously differentiable, so locally Lipschitz continuous. We can give a precise expression for the modulus of continuity in the case where the domain of definition is simple enough.

Proposition 2 (Modulus of continuity) *Let $\mathtt{sp} : \mathbb{R}_+ \to \mathbb{R}_+$, $f \in \mathrm{GVAL}[\mathtt{sp}]$. There exists $q \in \mathbb{K}[\mathbb{R}]$ such that for any $x_1, x_2 \in \operatorname{dom} f$, if $[x_1, x_2] \subseteq \operatorname{dom} f$ then $\|f(x_1) - f(x_2)\| \leqslant \|x_1 - x_2\| \, q(\mathtt{sp}(\max(\|x_1\|, \|x_2\|)))$. In particular, if $\operatorname{dom} f$ is convex then f has a polynomial modulus of continuity.*

Proof: Apply Definition 3 to get d, ℓ, n, p, x_0, y_0 and y. Let $k = \deg(p)$. Recall that for a matrix, the subordinate norm is given by $|||M||| = \max_i \sum_j |M_{ij}|$.

Then:

$$\begin{aligned}
\|f(x_1) - f(x_2)\| &= \left\| \int_{x_1}^{x_2} J_{y_{1..\ell}}(x) dx \right\| = \left\| \int_0^1 J_{y_{1..\ell}}((1-\alpha)x_1 + \alpha x_2)(x_2 - x_1) d\alpha \right\| \\
&\leqslant \int_0^1 |||J_{y_{1..\ell}}((1-\alpha)x_1 + \alpha x_2)||| \cdot \|x_2 - x_1\| \, d\alpha \\
&\leqslant \|x_2 - x_1\| \int_0^1 \max_{i \in [\![1,\ell]\!]} \sum_{j=1}^{d} |p_{ij}(y((1-\alpha)x_1 + \alpha x_2))| d\alpha \\
&\leqslant \|x_2 - x_1\| \int_0^1 \max_{i \in [\![1,\ell]\!]} \sum_{j=1}^{d} \Sigma p \max(1, \|y((1-\alpha)x_1 + \alpha x_2)\|)^k) d\alpha \\
&\leqslant \|x_2 - x_1\| \int_0^1 \max_{i \in [\![1,\ell]\!]} d\Sigma p \max(1, \mathtt{sp}(\|(1-\alpha)x_1 + \alpha x_2\|))^k d\alpha \\
&\leqslant \|x_2 - x_1\| \int_0^1 d\Sigma p \max(1, \mathtt{sp}(\max(\|x_1\|, \|x_2\|)))^k d\alpha \\
&\leqslant \|x_2 - x_1\| \, d\Sigma p \max(1, \mathtt{sp}(\max(\|x_1\|, \|x_2\|)))^k
\end{aligned}$$

□

4 Generable fields

In Section 2, we introduced the notion of *generable field*, which are fields with an additional stability property. We used this notion to ensure that the class of functions we built is closed under composition. It is well-known that if we allow any choice of constants in our computation, we will gain extra computational power because of uncomputable real numbers. For this reason, it is wise to make sure that we can exhibit at least one generable field consisting of computable real numbers only, and possibly only polynomial time computable numbers in the sense of computable analysis [BHW08].

Intuitively, we are looking for a (the) smallest generable field, call it $\mathbb{R}_G$, in order to minimize the computation power of the real numbers it contains. The rest of this section is dedicated to the study of this field. We first recall Definition 2.

Definition 4 (Generable field) *A field $\mathbb{K}$ is* generable *if and only if $\mathbb{Q} \subseteq \mathbb{K}$ and for any $\alpha \in \mathbb{K}$, and $(f : \mathbb{R} \to \mathbb{R}) \in \text{GVAL}_{\mathbb{K}}$, $f(\alpha) \in \mathbb{K}$.*

4.1 Extended stability

By definition of a generable field, $\mathbb{K}$ is preserved by unidimensional generable functions. An interesting question is whether $\mathbb{K}$ is also preserved by multidimensional functions. This is not immediate because because of several key differences in the definition of multidimensional generable functions. We first recall a folklore topology lemma.

Lemma 3 (Offset of a compact set) *Let $X \subseteq U \subseteq \mathbb{R}^n$ where U is open and X is compact. Then there exists $\varepsilon > 0$ such that $X_\varepsilon \subseteq U$ where the ε-offset of X is defined by $X_\varepsilon = \bigcup_{x\in X} B_\varepsilon(x)$.*

Proof:This is a very classical result: let $F = \mathbb{R}^n \setminus U$, then F is closed so the distance function[3] d_F to F is continuous. Since X is compact, $d_F(X)$ is a compact subset of $\mathbb{R}_+$, and $d_F(X)$ is nowhere 0 because $X \subseteq U \subseteq F$ where U is open. Consequently $d_F(X)$ admits a positive minimum ε. Let $x \in X_\varepsilon$, then $\exists y \in X$ such that $\|x - y\| < \varepsilon$, and by the triangle inequality, $\varepsilon \leqslant d_F(y) \leqslant \|x - y\| + d_F(x)$ so $d_F(x) > 0$ which means $x \notin F$, in other words $x \in U$. □

Lemma 4 (Polygonal path connectedness) *An open, connected subset U of $\mathbb{R}^n$ is always* polygonal-path-connected*: for any $a, b \in U$, there exists a polygonal path*[4] *from a to b in U. Furthermore, we can take all intermediate vertices in $\mathbb{Q}^n$.*

Proof:This is a textbook property, e.g. Theorem 3-5 in [?]. □

Proposition 3 (Generable path connectedness) *An open, connected subset U of $\mathbb{R}^n$ is always* generable-path-connected*: for any $a, b \in U \cap \mathbb{K}^n$, there exists $(\phi : \mathbb{R} \to U) \in \mathrm{GVAL}_\mathbb{K}$ such that $\phi(0) = a$ and $\phi(1) = b$.*

Proof: Let $a, b \in U \cap \mathbb{K}^n$ and apply Lemma 4 to get a polygonal path $\gamma : [0, 1] \to U$ from a to b. We are going to build a highly smoothed approximation of γ. This is usually done using bump functions but bump functions are not analytic, which complicates the matter. Furthermore, we need to build a path which domain of definition is $\mathbb{R}$, although this will be a minor annoyance only. We ignore the case where $a = b$ which is trivial and focus on the case where $a \neq b$.

Let $X = \gamma([0, 1])$ which is a compact connected set. Apply Lemma 3 to get $\varepsilon > 0$ such that $X_\varepsilon \subseteq U$. Without loss of generality, we can assume that $\varepsilon \in \mathbb{Q}$ so that it is generable.

Assume for a moment that γ is trivial, that is γ is a line segment from a to b. Let $\alpha \in \mathbb{N} \subseteq \mathbb{K}$ such that $\frac{1}{\tanh(\alpha)} \leqslant 1 + \frac{2\varepsilon}{\|b-a\|}$. It exists because $\frac{1}{\tanh(x)} \xrightarrow[x\to\infty]{} 1$. Define $\phi(t) = a + \frac{1+\mu(t)}{2}(b - a)$ where $\mu(t) = \frac{\tanh((2t-1)\alpha)}{\tanh(\alpha)}$. One can check that μ is an increasing function and that $\mu(0) = -1$ and $\mu(1) = 1$. Furthermore, if $t > 1$, $|\mu(t) - 1| < \frac{2\varepsilon}{\|b-a\|}$, and conversely, if $t < 0$, $|\mu(t) + 1| < \frac{2\varepsilon}{\|b-a\|}$. Consequently, $\phi(0) = a$, $\phi(1) = b$ and $\phi([0, 1])$ is the line segment between a and b, so $\phi([0, 1]) \subseteq X$. Furthermore, if $t < 0$, $\|a - \phi(t)\| \leqslant \left|\frac{1+\mu(t)}{2}\right| \|b - a\| < \varepsilon$,

[3] We always use the infinite norm $\|\cdot\|$ in this paper but it works for any distance

[4] A polygonal path is a connected sequence of line segments

and if $t > 1$, $\|b - \phi(t)\| \leqslant \left|\frac{1-\mu(t)}{2}\right| \|b - a\| < \varepsilon$. We conclude from this analysis that $\phi(\mathbb{R}) \subseteq X_\varepsilon \subseteq U$. It remains to show that $\phi \in \mathrm{GVAL}_\mathbb{K}$. Using Lemma 1, it suffices to show that $\tanh \in \mathrm{GVAL}_\mathbb{K}$ and $\frac{1}{\tanh(\alpha)} \in \mathbb{K}$. Since $\mathbb{K}$ is a field, we need to show that $\tanh(\alpha) \in \mathbb{K}$ which is a consequence of $\mathbb{K}$ being a generable field and tanh being a generable function. We already saw in Example 2 that $\tanh \in \mathrm{GVAL}_\mathbb{Q} \subseteq \mathrm{GVAL}_\mathbb{K}$.

In the general case where γ is a polygonal path, there are $0 = t_1 < t_2 < \ldots < t_k = 1$ such that $\gamma\restriction_{[t_i,t_{i+1}]}$ is the line segment between $x_i = \gamma(t_i)$ and $x_{i+1} = \gamma(t_{i+1})$, furthermore we can always take $x_i \in \mathbb{Q}^n$. Note that we can choose any parametrization for the path so in particular we can take $t_i = \frac{i}{k}$ and ensure that $t_i \in \mathbb{Q}$ for $i \in [\![0, k]\!]$. Since by hypothesis $x_0, x_n \in \mathbb{K}^n$, we get that $x_i \in \mathbb{K}^n$ and $t_i \in \mathbb{K}$ for all $i \in [\![0, k]\!]$.

Let us denote by $\phi_\varepsilon^{a,b}$ the path built in the previous case. We are simply going to add several instances of this path, with the necessary shifting and scaling. Since the errors will sum up, we will increase the approximation precision of each segment. Define $\phi(t) = a + \sum_{i=1}^{k-1} \left(\phi_{\varepsilon/k}^{x_i,x_{i+1}}\left(\frac{t-t_i}{t_{i+1}-t_i}\right) - x_i\right)$ and consider the following cases:

- if $t < 0$, then $\left\|\phi_{\varepsilon/k}^{x_i,x_{i+1}}\left(\frac{t-t_i}{t_{i+1}-t_i}\right) - x_i\right\| < \frac{\varepsilon}{k}$ for all $i \in [\![1, k-1]\!]$, so $\|a - \phi(t)\| < \frac{k-1}{k}\varepsilon$ and $\phi(t) \in X_\varepsilon$
- if $t \in [t_j, t_j + 1]$ for some j, then $\left\|\phi_{\varepsilon/k}^{x_i,x_{i+1}}\left(\frac{t-t_i}{t_{i+1}-t_i}\right) - x_i\right\| < \frac{\varepsilon}{k}$ for all $i > j$, and conversely $\left\|\phi_{\varepsilon/k}^{x_i,x_{i+1}}\left(\frac{t-t_i}{t_{i+1}-t_i}\right) - x_{i+1}\right\| < \frac{\varepsilon}{k}$ for all $i < j$. Finally $u = \phi_{\varepsilon/k}^{x_j,x_{j+1}}\left(\frac{t-t_j}{t_{j+1}-t_j}\right)$ belongs to the line segment from x_j to x_j+1. Since $a = x_1$, we get that $\|u - \phi(t)\| \leqslant \frac{k-1}{k}\varepsilon$ and thus $\phi(t) \in X_\varepsilon$.
- if $t > 1$ then $\|b - \phi(t)\| < \varepsilon$ for the same reason as $t < 0$, and thus $\phi(t) \in X_\varepsilon$.

We conclude that $\phi(\mathbb{R}) \subseteq X_\varepsilon \subseteq U$ and one easily checks that $\phi(0) = a$ and $\phi(1) = b$. Furthermore $\phi \in \mathrm{GVAL}_\mathbb{K}$ by Lemma 1 and because the x_i and t_i belong to $\mathbb{K}$ (see the details in the case of the trivial path). ☐

The immediate corollary of this result is that $\mathbb{K}$ is also preserved by multidimensional generable functions. Indeed, by composing a multidimensional function with a unidimensional one, we get back to the unidimensional case and conclude that any generable point in the input domain must have a generable image.

Corollary 2 (Generable field stability) *Let* $(f :\subseteq \mathbb{R}^d \to \mathbb{R}^\ell) \in \mathrm{GVAL}_\mathbb{K}$, *then* $f(\mathbb{K}^d \cap \operatorname{dom} f) \subseteq \mathbb{K}^\ell$.

Proof: Apply Definition 3 to get $n \in \mathbb{N}$, $p \in M_{n,d}(\mathbb{K})[\mathbb{R}^n]$, $x_0 \in \operatorname{dom} f \cap \mathbb{K}^d$, $y_0 \in \mathbb{K}^n$ and $y : \operatorname{dom} f \to \mathbb{R}^n$. Let $u \in \operatorname{dom} f \cap \mathbb{K}^d$. Since $\operatorname{dom} f$ is open and connected, by Proposition 3, there exists $(\gamma : \mathbb{R} \to \operatorname{dom} f) \in \mathrm{GVAL}$ such that

$\gamma(0) = x_0$ and $\gamma(1) = u$. Apply Definition 3 to γ to get $\bar{n} \in \mathbb{N}$, $\bar{p} \in M_{\bar{n},1}(\mathbb{K})[\mathbb{R}^{\bar{n}}]$, $\bar{x}_0 \in \mathbb{K}$, $\bar{y}_0 \in \mathbb{K}^{\bar{n}}$ and $\bar{y} : \mathbb{R} \to \mathbb{R}^{\bar{n}}$. Define $z(t) = y(\gamma(t)) = y(\bar{y}_{1..d}(t))$, then $z'(t) = J_y(\gamma(t))\gamma'(t) = p(y(\gamma(t)))\gamma'(t) = p(z(t))\bar{p}_{1..d}(\bar{y}(t))$ and $z(0) = y(\gamma(0)) = y(x_0) = y_0$. In other words $(\bar{y}, z)$ satisfy:

$$\begin{cases} \bar{y}(0) = x_0 \in \mathbb{K}^d \\ z(0) = y_0 \in \mathbb{K}^n \end{cases} \qquad \begin{cases} \bar{y}' = \bar{p}(\bar{y}) \\ z' = p(z)\bar{p}_{1..\ell}(\bar{y}) \end{cases}$$

Consequently $(z : \mathbb{R} \to \mathbb{R}^\ell) \in \text{GVAL}$ so, by definition of a generable field, $z(\mathbb{K}) \subseteq \mathbb{K}^z ell$. Conclude by noticing that $z(1) = y(\gamma(1)) = y(u)$. $\square$

4.2 Generable real numbers

In this section, we formalize the notion of generable field with an operator and study its properties. Recall that the smallest field we are looking for is a subset of $\mathbb{R}$ but it must also contains $\mathbb{Q}$. We consider the following operator G on subset of real numbers.

$$G : \begin{cases} \mathcal{P}(\mathbb{R}) & \to & \mathcal{P}(\mathbb{R}) \\ X & \mapsto & \displaystyle\bigcup_{f \in \text{GVAL}_X} f(X) \end{cases}$$

Comment 10 (G monotone and non-decreasing) *One can check that G is monotone ($X \subseteq G(X)$ for any $X \subseteq \mathbb{R}$). Indeed for any $x \in X$, the constant function $u \mapsto x$ belongs to GVAL_X. Moreover, it is non-decreasing because $\text{GVAL}_X \subseteq \text{GVAL}_Y$ if $X \subseteq Y$.*

It is clear that by definition, a field is generable if and only if it is G-stable. An interesting property of G is that its definition can be simplified. More precisely, by rescaling the functions, we can always assume that the image of G is produced by the evaluation of generable functions at a particular point, say 1, instead of the entire field.

Lemma 5 (Alternative definition of G) *If X is a field then,*

$$G(X) = \left\{ f(1) : f \in \text{GVAL}_X \right\}$$

Proof: Let $x \in G(X)$, then there exists $f \in \text{GVAL}_X$ and $t \in X$ such that $x = f(t)$. Consequently there exists $d \in \mathbb{N}$, $y_0 \in X^d$, $p \in X^d[\mathbb{R}^d]$ and $y : \mathbb{R} \to \mathbb{R}^d$ satisfying Definition 1:

- $y' = p(y)$ and $y(0) = y_0$
- $y_1 = f$

Consider $g(u) = f(ut)$ and note that $g(1) = f(t) = x$. We will see that $g \in \text{GVAL}_X$. Indeed, consider $z(u) = y(tu)$ then for all $u \in \mathbb{R}$:

- $z(0) = y(0) = y_0 \in X^d$;

- $z'(u) = ty'(tu) = tp(z(u)) = q(z(u))$ where $q = tp$ is a polynomial with coefficients in X since $t \in X$ and X is a field
- $z_1(u) = y_1(tu) = g(u)$

□

A consequence of this alternative definition is a simple proof that G preserves the property of being a field. This will turn out to be crucial fact later on.

Lemma 6 (G maps fields to fields) *If X is a field, then $G(X)$ is a field.*

Proof: Let $x, y \in G(X)$, by Lemma 5 there exists $f, g \in \mathrm{GVAL}_X$ such that $x = f(1)$ and $y = g(1)$. Apply Lemma 1 to get that $f \pm g$ and fg belong to GVAL_X And thus $x \pm y$ and xy belong to $G(X)$.

Finally the case of $\frac{1}{x}$ (when $x \neq 0$) is slightly more subtle: we cannot simply compute $\frac{1}{f}$ because f may cancel. Instead we are going to compute $\frac{1}{g}$ where $g(1) = f(1)$ but g nevers cancels.

First, note that we can always assume that $x > 0$ because $G(X)$ is closed under the negation, and $-\frac{1}{x} = \frac{1}{-x}$. Since $f(1) = x > 0$ and f is continuous, it means there exists $\varepsilon > 0$ such that $f(t) > 0$ for all $t \in [1-\varepsilon, 1+\varepsilon]$ and we can take $\varepsilon \in \mathbb{Q}$. Define $g(t) = f(t) + \left(1 + f(t)^2\right)\left(\frac{t-1}{\varepsilon}\right)^2$. It is not hard to see that $g(1) = f(1)$ and that $g(t) > 0$ for all $t \in \mathbb{R}$. Furthermore, $g \in \mathrm{GVAL}_X$ because of Lemma 1. Note that we use the part of the lemma which does not assume that X is a generable field!

Using Lemma 1, we conclude that $\frac{1}{g} \in \mathrm{GVAL}_X$ and thus $\frac{1}{x} \in G(X)$. □

Not only G maps fields to fields, but it also preserves polynomial-time computability. This is of major interest to us to show that there exists a generable field with low complexity numbers. Here $\mathbb{R}_P$ denotes the set of polynomial time computable real numbers [Ko91].

Lemma 7 (G preserves polytime computability) *G maps subsets of polynomial time computable real numbers into themselves,* i.e. *for any $X \subseteq \mathbb{R}_P$, $G(X) \subseteq \mathbb{R}_P$.*

Proof: Let $X \subseteq \mathbb{R}_P$ and $x \in G(X)$, $f \in \mathrm{GVAL}_X$ and $t \in X$ such that $x = f(t)$. We can use [**?**] to conclude that x is polynomial time computable, thus $x \in \mathbb{R}_P$. □

Finally, the core of what makes G very special is its finiteness property. Essentially, it means that if $x \in G(X)$ then x really only requires a finite number of elements in X to be computed. In the framework of order and lattice theory, this shows that G is a Scott-continuous function between the complete partial order (CPO) $(\mathcal{L}, \subseteq)$ and itself.

Lemma 8 (Finiteness of G) *For any $X \subseteq \mathbb{R}$ and $x \in G(X)$, there exists a finite $Y \subseteq X$ such that $x \in G(Y)$.*

Proof: Let $x \in G(X)$, then there exists $f \in \text{GVAL}_X$ and $t \in X$ such that $x = f(t)$. Then there exists $y_0 \in X^d$ and a polynomial p with coefficients in X such that f satisfies Definition 1. Define Y as the subset of X containing t, the components of y_0 and all the coefficients of p. Then Y is finite and $f \in \text{GVAL}_Y$. Furthermore $t \in Y$ so $x \in G(Y)$. □

We can now define the set of "generable real numbers", call it $\mathbb{R}_G$. The main result of this section is that $\mathbb{R}_G$ is the smallest generable field. But more surprisingly, we show that all the elements of $\mathbb{R}_G$ are polynomial time computable (in the sense of Computable Analysis).

Definition 5 (Generable real numbers)

$$\mathbb{R}_G = \bigcup_{n \geqslant 0} G^{[n]}(\mathbb{Q}).$$

Theorem 2 ($\mathbb{R}_G$ is generable subfield of $\mathbb{R}_P$) *$\mathbb{R}_G$ is the smallest generable field for inclusion. Furthermore, it form a generable subfield of polynomial time computable real numbers in the sense of Computable Analysis, i.e. $\mathbb{R}_G \subseteq \mathbb{R}_P$.*

Proof: First observe that any generable field must contain $\mathbb{R}_G$. Indeed, let $\mathbb{K}$ be a generable field: then $G(\mathbb{K}) \subseteq \mathbb{K}$ by definition. But G is non-decreasing thus $G(\mathbb{Q}) \subseteq G(\mathbb{K}) \subseteq \mathbb{K}$. By applying G repeatedly, we get that $G^{[n]}(\mathbb{Q}) \subseteq \mathbb{K}$ for all n. Thus $\mathbb{R}_G \subseteq \mathbb{K}$.

Conversely, we need to show that $\mathbb{R}_G$ is a field. Observe that since G is monotone, $G^{[n]}(\mathbb{Q})$ is an increasing sequence (for inclusion). Let $x, y \in \mathbb{R}_G$, then there exists $n \in \mathbb{N}$ such that $x, y \in G^{[n]}(\mathbb{Q})$. Apply Lemma 6 to get that $G^{[n]}(\mathbb{Q})$ is a field. It follows that $x + y, x - y, xy$ and $\frac{x}{y}$ (if $y \neq 0$) belong to $G^{[n]}(\mathbb{Q}) \subseteq \mathbb{R}_G$. Thus $\mathbb{R}_G$ is a field.

It remains to show that $\mathbb{R}_G$ is a *generable* field. This follows from Lemma 8: let $x \in G(\mathbb{R}_G)$, then there exists a **finite** $Y \subseteq \mathbb{R}_G$ such that $x \in G(Y)$. Using the same reasoning as above, there exists $n \in \mathbb{N}$ such that $Y \subseteq G^{[n]}(\mathbb{Q})$. Thus $x \in G(Y) \subseteq G(G^{[n]}(\mathbb{Q})) = G^{[n+1]}(\mathbb{Q}) \subseteq \mathbb{R}_G$. It follows that $G(\mathbb{R}_G) \subseteq \mathbb{R}_G$, *i.e.* it is generable.

Finally, since $\mathbb{Q} \subseteq \mathbb{R}_P$, iterating Lemma 7 yields that $G^{[n]}(\mathbb{Q}) \subseteq \mathbb{R}_P$ for all $n \in \mathbb{N}$ and thus $\mathbb{R}_G \subseteq \mathbb{R}_P$. □

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