In this note, we generalize the polynomial bound on convergence time of averaging algorithms established by Nedic et al. [2] in the case of doubly stochastic matrices. Our contribution consists in observing that the arguments developed in [2] provide an upper bound on the singular values of general (non doubly) stochastic matrices.

Interestingly, our polynomial bound also applies to the fixed-weight algorithm with a time-varying bidirectional topology. It thus unifies the time complexity results in [2], [1] and in [3].

1 Singular values of a stochastic matrix

1.1 Preliminaries

Let \( n \) be a positive integer and let \([n] = \{1, \ldots, n\}\).

Let \( \pi \in \mathbb{R}^n \) be a positive probability vector in \( \mathbb{R}^n \). We define

\[
<x, y>_{\pi} = \sum_{i=1}^{n} \pi_i x_i y_i
\]

that is an inner product on \( \mathbb{R}^n \). For any \( n \times n \) square matrix \( A \), \( A^{\dagger}_\pi \) denotes \( A \)'s adjoint for the inner product \( <\cdot, \cdot>_{\pi} \). We easily check that

\[
A^{\dagger}_\pi = \frac{\pi_j}{\pi_i} A_{ji}.
\]

In the case \( A \) is an ergodic matrix, its Perron vector denoted \( \pi(A) \) is a positive vector. For simplicity, we write \( A^\dagger \) instead of \( A^{\dagger}_{\pi(A)} \). We easily check that if \( A \) is a stochastic matrix, then \( A^\dagger \) is also a stochastic matrix with the same Perron vector, i.e., \( \pi(A^\dagger) = \pi(A) \). Therefore \( A^\dagger A \) is self-adjoint for the inner product \( <\cdot, \cdot>_{\pi(A)} \).

1.2 A formula à la Green

We start with an equality that is a generalization of the Green’s formula.

**Proposition 1.** If \( \pi \) is a positive probability vector in \( \mathbb{R}^n \) and \( L \) is a square matrix of size \( n \) such that \( 1 \in \ker(L) \) and \( L^{\dagger}_{\pi} = L \), then for all vector \( x \in \mathbb{R}^n \), we have

\[
<x, Lx>_{\pi} = -\frac{1}{2} \sum_{(i,j) \in [n]^2} \pi_i L_{i,j} (x_i - x_j)^2.
\]
Proof. First we observe that
\[
\sum_{(i,j) \in [n]^2} \pi_i L_{ij} (x_i - x_j)^2 = \sum_{i \neq j} \pi_i L_{ij} (x_i - x_j)^2
\]
\[
= \sum_{i \neq j} \pi_i L_{ij} x_i^2 + \sum_{i \neq j} \pi_i L_{ij} x_j^2 - 2 \sum_{i \neq j} \pi_i L_{ij} x_i x_j.
\]
Because of the properties of \( L \), the first two terms are both equal to \(- \sum_{i=1}^{n} \pi_i L_{ii} x_i^2 \) and so
\[
\sum_{(i,j) \in [n]^2} \pi_i L_{ij} (x_i - x_j)^2 = -2 \left( \sum_{i=1}^{n} \pi_i L_{ii} x_i^2 + \sum_{i \neq j} \pi_i L_{ij} x_i x_j \right).
\]
Besides, we have
\[
\langle x, Lx \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i L_{ij} x_i x_j = \sum_{i=1}^{n} \pi_i L_{ii} x_i^2 + \sum_{i \neq j} \pi_i L_{ij} x_i x_j
\]
and the lemma then follows. \( \square \)

We immediately derive the following corollary for ergodic stochastic matrices.

**Corollary 2.** Let \( A \) be an ergodic stochastic matrix of size \( n \) and let \( \pi \) denote its Perron vector. For all vector \( x \in \mathbb{R}^n \),
\[
\langle x, x \rangle_{\pi} - \langle x, A^T A . x \rangle_{\pi} = \frac{1}{2} \sum_{(i,j) \in [n]^2} \pi_i (A^T A)_{i,j} (x_i - x_j)^2.
\]

As an immediate consequence of the above corollary, we obtain that the restriction of the quadratic form
\[
Q_A(x) = \langle x, x - A^T A . x \rangle_{\pi}
\]
to the orthogonal complement of \( \mathbb{R} \mathbf{1} \) in \( \mathbb{R}^n \) is positive definite.

### 1.3 An upper bound on the singular values of a stochastic matrix

Let \( A \) be an ergodic stochastic matrix of size \( n \) with positive diagonal entries. Since \( A^T A \) is a stochastic matrix, the \( n \) singular values of \( A \) (which are the square roots of \( A^T A \)) in the increasing order satisfy \( 0 \leq \sigma_n \leq \ldots \leq \sigma_2 \leq \sigma_1 = 1 \). By the Perron-Frobenius theorem, we have
\[
\sigma_2 < 1.
\]
The aim of the section is to refine the latter inequality.

First we fix some notation: the Perron vector of \( A \) is denoted by \( \pi \) and we let
\[
\mu_A = \min \{ \pi_i A_{ij} \mid A_{ij} > 0 \}.
\]
Let \( \Delta \) be the real vector space generated by \( \mathbf{1} = (1, \ldots, 1)^T \), and \( \Delta^\perp \) be the orthogonal complement of \( \Delta \) in \( \mathbb{R}^n \). We denote by \( \delta \) the semi-norm on \( \mathbb{R}^n \) defined by
\[
\delta(x) = \max_{i=1,\ldots,n} (x_i) - \min_{i=1,\ldots,n} (x_i).
\]
which is a norm on $\Delta^\perp$.

Moreover, let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be an orthonormal basis in which $A^\dagger A$ is diagonalizable and such that $A^\dagger A\varepsilon_i = \sigma_i^2\varepsilon_i$ and $\varepsilon_1 = 1$. We start by two lemmas which are both slight variations of two results established in [2].

**Lemma 3** (Lemma 5 in [2]). Let $N_1 \cup N_2$ a partition of $[n]$ into two disjoint sets. If there exist two indices $i \in N_1$ and $j \in N_2$ such that $A_{ij} > 0$, then

$$\sum_{i \in N_1, j \in N_2} \pi_i (A^\dagger A)_{ij} \geq \mu_A/2.$$ 

**Proof.** First we observe that

$$\pi_i (A^\dagger A)_{ij} = \sum_{k \in N} \pi_k A_{ki} A_{kj}.$$ 

Moreover, since $A$ is a stochastic matrix, for every index $i$, we have either $\sum_{j \in N_2} A_{ij} \geq 1/2$ or $\sum_{j \in N_1} A_{ij} \geq 1/2$. That leads us to consider the two following cases.

1. There exists an index $i^* \in N_1$ such that $\sum_{j \in N_2} A_{i^*j} \geq 1/2$. Then we have

$$\sum_{i \in N_1, j \in N_2} \pi_i (A^\dagger A)_{ij} \geq \sum_{j \in N_2} \sum_{i \in N_1} \pi_i A_{i^*i} A_{i^*j} \geq \frac{\pi_{i^*} A_{i^*i}}{2}.$$ 

2. Otherwise for each index $i \in N_1$, we have $\sum_{j \in N_1} A_{ij} \geq 1/2$. It follows that

$$\sum_{i \in N_1, j \in N_2} \pi_i (A^\dagger A)_{ij} \geq \sum_{i \in N_1, j \in N_2} \sum_{k \in N_1} \pi_k A_{ki} A_{kj} = \sum_{k \in N_1, j \in N_2} \pi_k A_{kj} \sum_{i \in N_1} A_{ki}.$$ 

Hence we have

$$\sum_{i \in N_1, j \in N_2} \pi_i (A^\dagger A)_{ij} \geq \frac{1}{2} \sum_{i \in N_1, j \in N_2} \pi_i A_{ij}.$$ 

By assumption, there exist $k_1 \in N_1$ and $k_2 \in N_2$ such that $A_{k_1k_2} > 0$ and

$$\sum_{j \in N_1, j \in N_2} \pi_j (A^\dagger A)_{ij} \geq \frac{\pi_{k_1} A_{k_1k_2}}{2}.$$ 

In both cases, we then derive that $\sum_{i \in N_1, j \in N_2} \pi_i (A^\dagger A)_{ij} \geq \mu_A/2$. 

**Lemma 4** (Lemma 8 in [2]). For every vector $x \in \mathbb{R}^n$, we have

$$Q_A(x) \geq \frac{\mu_A}{2n} (\delta(x))^2.$$ 

**Proof.** Using index permutation, we assume that $x_1 \leq \ldots \leq x_n$. Since for any nonnegative numbers $v_1, \ldots, v_k$, we have

$$(v_1 + \cdots + v_k)^2 \geq v_1^2 + \cdots + v_k^2,$$

it follows that

$$\sum_{i<j} \pi_i (A^\dagger A)_{ij} (x_i - x_j)^2 \geq \sum_{i<j} \pi_i (A^\dagger A)_{ij} \sum_{d=i}^{j-1} (x_{d+1} - x_d)^2.$$
By reordering the terms in the last sum, we obtain
\[
\sum_{i<j} \pi_i (A^\dagger A)_{ij} (x_i - x_j)^2 \geq \sum_{d=1}^{n-1} \sum_{i=1}^d \sum_{j=d+1}^n \pi_i (A^\dagger A)_{ij} (x_{d+1} - x_d)^2.
\]
Then we use Lemma 3 to show that for each \(d \in \{1, \ldots, n-1\}\), we have
\[
\sum_{i=1}^d \sum_{j=d+1}^n \pi_i (A^\dagger A)_{ij} \geq \mu_A/2.
\]
Therefore
\[
Q_A(x) \geq \frac{\mu_A}{2} \sum_{d=1}^{n-1} (x_{d+1} - x_d)^2.
\]
By Cauchy-Schwarz, we obtain
\[
\sum_{d=1}^{n-1} (x_{d+1} - x_d)^2 \geq \frac{1}{n} (x_n - x_1)^2,
\]
which completes the proof.

Lemma 5. Each vector in the orthonormal basis \(\{\varepsilon_2, \ldots, \varepsilon_n\}\) of \(\Delta^\perp\) has a norm \(\delta\) greater than 1, i.e.,
\[
\forall i \in \{2, \ldots, n\}, \delta(\varepsilon_i) > 1.
\]
Proof. We denote by \(\{e_1, \ldots, e_n\}\) the standard basis and we let
\[
\varepsilon_i = u_{i,1} e_1 + \cdots + u_{i,n} e_n.
\]
By definition of the inner product \(\langle \cdot, \cdot \rangle_\pi\), we have \(\|e_k\|^2_\pi = \pi_k\) and \(\langle e_k, e_\ell \rangle_\pi = 0\) when \(k \neq \ell\). It follows that for every \(i \in \{2, \ldots, n\}\),
\[
\|\varepsilon_i\|^2_\pi = u_{i,1}^2 \pi_1 + \cdots + u_{i,n}^2 \pi_n = 1
\]
and
\[
\langle \varepsilon_i, e_1 \rangle_\pi = \langle \varepsilon_i, 1 \rangle_\pi = u_{i,1} \pi_1 + \cdots + u_{i,n} \pi_n = 0.
\]
From these equalities, we derive that there exist two different indices \(k_i\) and \(\ell_i\) such that
\[
|u_{k_i}| > 1 \quad \text{and} \quad u_{k_i}u_{\ell_i} < 0,
\]
which implies
\[
\delta(\varepsilon_i) > 1.
\]

Proposition 6. If \(A\) is an ergodic stochastic matrix of size \(n\) with positive diagonal entries, then each singular value of \(A\) different from 1 is at most equal to \(\sqrt{1 - \frac{\mu_A}{2n}}\).
Proof. We easily check that
\[ Q_A(\varepsilon_2) = 1 - (\sigma_2)^2. \]
Besides combining Lemmas 4 and 5, we obtain
\[ Q_A(\varepsilon_2) > \frac{\mu_A}{2n}, \]
which shows the upper bound on \( \sigma_2. \)

1.4 Simpler proof and improvement in the case of a self-adjoint matrix
In the case \( A \) is a self-adjoint ergodic matrix, a simplified version of the above proof, in which Lemmas 4 and 5 are omitted, proves that any \( A \)'s eigenvalue \( \lambda \) different from 1 satisfies
\[ \lambda < 1 - \frac{\mu_A}{2D_A}, \tag{1} \]
where \( D_A \) denotes the diameter of the graph associated to \( A \); see Chazelle [1].
Moreover \( \lambda \) lies within at least one Gershgorin disc \( D(A_{ii}, 1 - A_{ii}), \) i.e.,
\[ -1 + 2A_{ii} \leq \lambda \leq 1. \]
It then follows that
\[ |\lambda| < \max\{1 - \frac{\mu_A}{2D_A}, 1 - 2\alpha_A\}, \]
where \( \alpha_A = \min_{i=1,...,n} (A_{ii}). \) We easily check that \( \mu_A \leq \alpha_A, \) and thus
\[ |\lambda| < 1 - \frac{\mu_A}{2D_A}. \]
If \( \lambda_1(A), \ldots, \lambda_n(A) \) denote \( A \)'s eigenvalue ranged in the order of magnitude, then \( \lambda_1(A) = 1, \) and the second Weyl inequality gives
\[ \sigma_2(A) \leq \lambda_2(A). \]
The upper bound in Proposition 6 then can be easily improved into
\[ \sigma_2(A) \leq 1 - \frac{\mu_A}{2n}. \]

2 A convergence theorem and its applications
We consider a sequence \( (A_t)_{t \in \mathbb{N}} \) of \( n \times n \) matrices that satisfies the following assumptions:
A1: Every matrix \( A_t \) is an ergodic stochastic matrix with a positive diagonal.
A2: All the matrices \( A_t \) have the same Perron vector denoted by \( \pi: \)
\[ \forall t \in \mathbb{N}, \quad \pi(A_t) = \pi. \]
A3: There exists some positive lower bound \( \alpha \) on the positive entries of the matrices \( A_t:\)
\[ \forall t \in \mathbb{N}, \forall (i,j) \in \{1,\ldots,n\}^2, \quad (A_t)_{ij} \in \{0\} \cup [\alpha,1]. \]
Let \( x(0) \in \mathbb{R}^n \), and let \( (x(t))_{t \in \mathbb{N}} \) denote the sequence of vectors in \( \mathbb{R}^n \) defined by:

\[
x(t + 1) = A_t x(t).
\]

**Theorem 7.** Under assumptions A1-3, the sequence \( (x(t))_{t \in \mathbb{N}} \) converges to a vector \( x^* \) that is colinear to \( 1 \) and

\[
\lim_{t \to +\infty} \| x(t) - x^* \|^{1/t} \leq 1 - \frac{\mu}{4n},
\]

where \( \mu = \inf \{ \pi_i(A_t)_{ij} : (A_t)_{ij} > 0 \} \).

**Proof.** By (2), we have:

\[
\langle x(t), 1 \rangle_{\pi} = \langle A_{t-1} x(t-1), 1 \rangle_{\pi} = \langle x(t-1), A_{t-1}^\dagger 1 \rangle_{\pi}.
\]

Since \( A_{t-1}^\dagger \) is a stochastic matrix, \( A_{t-1}^\dagger 1 = 1 \) and so

\[
\langle x(t), 1 \rangle_{\pi} = \langle x(t-1), 1 \rangle_{\pi}.
\]

Therefore, the orthogonal projection of \( x(t) \) on \( \Delta \) is constant.

We let \( a = \langle x(0), 1 \rangle_{\pi} \) and

\[
V(t) = \| x(t) - a1 \|^2.
\]

Then

\[
V(t) = \| x(t) \|^2 - 2a \langle x(t), 1 \rangle_{\pi} + a^2 \| 1 \|^2 = \| x(t) \|^2 - a^2,
\]

and

\[
V(t) - V(t + 1) = \langle x(t), x(t) \rangle_{\pi} - \langle A_t x(t), A_t x(t) \rangle_{\pi} = \langle x(t), x(t) - A_t^\dagger A_t x(t) \rangle_{\pi}.
\]

By Corollary 2, it follows that \( V \) is non-increasing; we shall prove that \( V(t) \) tends to 0.

By Proposition 6, \( A_t \) has \( n \) real singular values \( \sigma_1, \ldots, \sigma_n \) that satisfy

\[
0 \leq \sigma_n \leq \ldots \sigma_2 \leq \sqrt{1 - \frac{\mu}{2n}} < \sigma_1 = 1
\]

with \( \mu = \min \{ \pi_i(A_t)_{ij} : (i, j) \in E(G_t) \land t \in \mathbb{N} \} \). Let \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) be an orthonormal basis for the inner product \( \langle \cdot, \cdot \rangle_{\pi} \) such that for each index \( i \in [n] \),

\[
A_t^\dagger A_t \varepsilon_i = \sigma_i^2 \varepsilon_i.
\]

In particular, \( \varepsilon_1 = 1 \).

Let \( z_1, \ldots, z_n \) the components of \( x(t) \) in this basis\(^1\), namely,

\[
x(t) = z_1 \varepsilon_1 + \cdots + z_n \varepsilon_n.
\]

Hence \( z_1 = \langle x(0), 1 \rangle_{\pi} \) and so

\[
V(t) = z_2^2 + \cdots + z_n^2.
\]

Moreover, we have

\[
A_t^\dagger A_t x(t) = z_1 \varepsilon_1 + z_2 \sigma_2^2 \varepsilon_2 + \cdots + z_n \sigma_n^2 \varepsilon_n.
\]

\(^1\)The real numbers \( z_i, \lambda_i, \) and the vectors \( \varepsilon_i \) all depend on \( t \), but \( t \) is not explicit in our notation as no confusion can arise.
and thus
\[ V(t) - V(t + 1) = z_2^2(1 - \sigma_2^2) + \cdots + z_n^2(1 - \sigma_n^2). \]
Hence
\[ V(t) - V(t + 1) \geq (1 - \sigma_2^2)V(t). \]
From the upper bound on the second singular value in Proposition 6, it follows that
\[ V(t) \leq \left(1 - \frac{\mu}{2n}\right)^t V(0). \]
Hence \( \lim_{t \to \infty} V(t) = 0 \), and so
\[ \lim_{t \to \infty} x(t) = a \mathbf{1}. \]
To complete the proof, we use the inequality \((1 - u)^{1/2} \leq 1 - u/2\) which holds for any \(u \in [0,1]\).

We now consider a system with \(n\) agents \(\{1, \ldots, n\}\), a local variable \(x_i\) for each agent \(i\), and an averaging algorithm \(A\) with with non-vanishing and bounded weights, i.e., if \(G_t\) is the communication graph at round \(t\) and \(x_i(t)\) is the value of \(x_i\) at the end of round \(t\):
\[ x_i(t) = \sum_{k \in \mathcal{N}_i(G_t)} w_{ik}(t)x_k(t - 1), \quad (3) \]
with a positive lower bound on all the weights \(w_{ik}(t)\).

We consider an execution of \(A\) with an initial state \(x(0) \in \mathbb{R}^n\) and a communication pattern (sequence of communication graphs) \((G_t)_{t \in \mathbb{N}}\). We say that \(A\) achieves asymptotic consensus in this execution if the sequence \(x(t)\) converges to a vector \(x^*\) that is colinear to \(\mathbf{1}\). The convergence rate is then defined as
\[ \varrho = \lim_{t \to \infty} \|x(t) - x^*\|^{1/t} \]
where \(\|\cdot\|\) is any norm on \(\mathbb{R}^n\), and the convergence time is
\[ T(\varepsilon) = \inf\{\tau \in \mathbb{N} : \forall t \geq \tau, \ V(t) \leq \varepsilon V(0)\}. \]

Let \(A_t\) denote the stochastic matrix associated to the update rule (3) at round \(t\) in this execution. The central assumption is that all the stochastic matrices \(A_t\) share the same Perron vector \(\pi\):
\[ \forall t \in \mathbb{N}, \quad \pi(A_t) = \pi. \]
We denote
\[ \mu = \inf \{\pi_i(A_t)_{ij} : (A_t)_{ij} > 0\}. \]
From Theorem 7, we immediately deduce the following corollary.

**Corollary 8.** The algorithm \(A\) achieves asymptotic consensus with convergence rate \(\varrho \leq 1 - \frac{\mu}{4n}\) and convergence time \(T(\varepsilon) \leq \frac{2n}{\mu} \log \left(1/\varepsilon\right)\).

In the reversible case where every matrix \(A_t\) is self-adjoint, the remark in Section 1.4 provides a better upper bound on the second singular value of \(A_t\), and leads to the following improvements on the convergence rate and convergence time.

**Corollary 9.** In the reversible case, the algorithm \(A\) achieves asymptotic consensus with convergence rate \(\varrho \leq 1 - \frac{\mu}{2n}\) and convergence time \(T(\varepsilon) \leq \frac{n}{\mu} \log \left(1/\varepsilon\right)\).
2.1 Applications and previous results

Corollary 8 has several applications to (1) the EqualNeighbor algorithm with a fixed bidirectional topology, (2) the FixedWeight algorithm with a dynamic bidirectional topology, and finally the Metropolis algorithm with a dynamic bidirectional topology.

2.1.1 EqualNeighbor algorithm with a fixed bidirectional topology

With the EqualNeighbor algorithm, we have

\[ w_{ik}(t) = \frac{1}{d_i(t)}, \]

where \( d_i(t) \) is the number of i’s in-neighbors in \( G_t \). If \( G_t \) is bidirectional, then the i-th entry of the Perron vector of \( A_t \) is equal to:

\[ \pi_i = \frac{d_i(t)}{d(t)}, \]

where \( d(t) = \sum_{i=1}^{n} d_i(t) \). Therefore \( \mu \geq 1/n^2 \). Moreover, the stochastic matrix \( A \) associated in each round is self-adjoint.

Corollary 9 then applies to any execution of the EqualNeighbor algorithm with a fixed bidirectional topology.

**Theorem 10** ([3]). In any execution with a fixed bidirectional connected topology, the EqualNeighbor algorithm achieves asymptotic consensus with convergence rate \( \varrho \leq 1 - \frac{1}{2n^2} \) and convergence time \( T(\varepsilon) \leq n^3 \log \left( \frac{1}{\varepsilon} \right) \).

2.1.2 FixedWeight algorithm with a dynamic bidirectional topology

For each agent \( i \), let \( q_i \) denote an upper bound on the number of i’s in-neighbors in a given communication pattern \( (G_t) \). Weights in the FixedWeight algorithm are given by:

\[ w_{ik}(t) = \begin{cases} 
\frac{1}{q_i} & \text{if } j \in N_i^+(t) \setminus \{i\} \\
1 - (d_i(t) - 1)/q_i & \text{if } j = i \\
0 & \text{otherwise}
\end{cases} \]

We easily check that if \( G_t \) is bidirectional, then the i-th entry of the Perron vector of \( A_t \) is equal to:

\[ \pi_i(A_t) = \frac{q_i}{q}, \]

where \( q = \sum_{i=1}^{n} q_i \).

It then follows that in any bidirectional communication pattern, the Perron vector is constant and \( \mu = 1/q \geq 1/n^2 \). Moreover, for every round \( t \), the stochastic matrix \( A_t \) is self-adjoint.

Corollary 9 then applies to any execution of the FixedWeight algorithm with a bidirectional communication pattern.

**Theorem 11** ([1]). In any execution with a communication pattern composed of bidirectional connected communication graphs, the FixedWeight algorithm achieves asymptotic consensus with convergence rate \( \varrho \leq 1 - \frac{1}{2n^2} \) and convergence time \( T(\varepsilon) \leq n^3 \log \left( \frac{1}{\varepsilon} \right) \).
2.1.3 Metropolis algorithm with a dynamic bidirectional topology

With the Metropolis algorithm, we have

\[
    w_{ik}(t) = \begin{cases} 
        \frac{1}{\max(d_i(t),d_j(t))} & \text{if } j \in N_i^+(t) \setminus \{i\} \\
        1 - \sum_{j \in N_i^+(t) \setminus \{i\}} \frac{1}{\max(d_i(t),d_j(t))} & \text{if } j = i \\
        0 & \text{otherwise .} 
    \end{cases}
\]

We easily check that if \( G_t \) is bidirectional, then each matrix \( A_t \) is symmetric, and so doubly stochastic. It follows that the Perron vector of \( A_t \) is collinear to 1, its \( i \)-th entry is equal to:

\[
    \pi_i(A_t) = \frac{1}{n},
\]

and \( \mu \geq 1/n^2 \). Moreover \( A_t \) is self-adjoint for the inner product \((.,.)_\pi\).

Corollary 9 then applies to any execution of the Metropolis algorithm with a bidirectional communication pattern.

**Theorem 12** ([2]). In any execution with a communication pattern composed of bidirectional connected communication graphs, the Metropolis algorithm achieves asymptotic consensus with convergence rate \( \varrho \leq 1 - \frac{1}{2n^2} \) and convergence time \( T(\varepsilon) \leq n^3 \log \left( \frac{1}{\varepsilon} \right) \).

**References**

