A library of wqo datatypes, part III (multisets, finite sets, etc.)

Application: van der Meyden’s magic algorithm in databases

A library of wqo datatypes, part IV: Kruskal’s theorem

Application: termination of rewrite systems
INTERMISSION:
A LIBRARY OF WQOS (III)

\[
D ::= A \quad \text{finite qos, lemma 1.13 (1)} \\
\mathbb{N} \quad \text{lemma 1.2} \\
\sum_{i=1}^{n} D_i \quad \text{finite disjoint sums, lemma 1.14} \\
\prod_{i=1}^{n} D_i \quad \text{finite products, corollary 1.16} \\
D^* \quad \text{finite words, embedding, theorem 1.19} \\
D^\circ \quad \text{finite multisets, embedding, corollary 1.21} \\
\mathcal{P}_f(D) \quad \text{finite sets, Hoare qo, Corollary 1.26} \\
T(D) \quad \text{finite trees, homeomorphic embedding, theorem 1.29} \\
\mathcal{G} \quad \text{finite unordered graphs, section 1.7}
\]
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EASY CONSTRUCTIONS

- **Lemma 1.13:**

  1. every subset of a wqo is wqo *(why?)*
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  3. If \(f:X \rightarrow Y\) surjective monotonic and \(\leq_X\) wqo then \(\leq_Y\) wqo.
Lemma 1.13:

1. every subset of a wqo is wqo (why?)

2. [extension] if $(\forall x, y, x \leq y \Rightarrow x \preceq' y)$ and $\leq$ wqo then $\leq'$ wqo (why?)

3. If $f : X \to Y$ surjective monotonic and $\leq_X$ wqo then $\leq_Y$ wqo.

Proof of 3: every infinite sequence in $Y$ is of the form $(f(x_n))_{n \in \mathbb{N}}$. Find $m < n$ such that $x_m \leq_X x_n$. Hence $f(x_m) \leq f(x_n)$. $\square$
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2. [extension] if \((\forall x,y, x \leq y \Rightarrow x \leq 'y)\) and \(\leq\) wqo then \(\leq '\) wqo (why?)

3. If \(f: X \to Y\) surjective monotonic and \(\leq_X\) wqo then \(\leq_Y\) wqo.

4. If \(f: X \to Y\) reflects order \((\forall x,y, f(x) \leq f(y) \Rightarrow x \leq y)\)

   and \(\leq_Y\) wqo then \(\leq_X\) wqo. [generalizes item 1]
A **multiset** over \( D = \text{map } D \rightarrow \mathbb{N} \) with finitely many \( \neq 0 \) entries

= « sets of elements with multiplicities »

= « lists, modulo permutation »

E.g., \(|a,a,b,b,b,c|\) = \{5\mapsto0, a\mapsto2, b\mapsto3, c\mapsto1, \ldots, z\mapsto0\}
MULTISETS

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- Let $D^{\circ}$ denote the set of all multisets over $D$. Write $\cup$ for multiset union.
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- **Parikh mapping** $\varphi : D^* \to D^\circ$, forgets order of elements
  
  E.g., $\varphi(\text{cabbab}) = \{|a,a,b,b,b,c|\}$
MULTISETS

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  Write $\cup$ for multiset union.

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  E.g., $\varphi(\text{cabbab}) = \{|a,a,b,b,b,c|\}$

- Define $\leq_\circ$ by

  \[
  \begin{align*}
  \emptyset \leq_\circ & m' (\emptyset) \\
  m & \leq_\circ m' \cup \{d\} (\text{add}) \\
  d & \leq d' \quad m & \leq_\circ m' (\text{inc})
  \end{align*}
  \]
MULTISETS

- $\varphi$ is monotonic and surjective.
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- Recall **Lemma 1.13** (3):
The image of a wqo by a surjective monotonic map is wqo.
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- \( \varphi \) is monotonic and surjective.

- Recall **Lemma 1.13** (3):
The image of a wqo by a surjective monotonic map is wqo.

- **Corl 1.21**: If \( \leq \) wqo on \( D \), then \( \leq \ominus \) wqo on \( D^\oplus \). \(\square\)
MULTISETS

- Don’t confuse \( \leq \) with \textbf{multiset extension} \( \leq_{\text{mul}} \):
m \( <_{\text{mul}} \) \( m' \) iff one obtains \( m \) from \( m' \) by replacing elements by (arbitrary many) strictly smaller elements, at least once.

- E.g., \(|3| >_{\text{mul}} |2,2,2| >_{\text{mul}} |2,2| >_{\text{mul}} |1,1,\ldots,1,2|\)
MULTISETS

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- **Thm** (Dershowitz, Manna 1979): if $< \text{wf}$ then $<_{\text{mul}} \text{wf}$. *(Proof sketch: see footnote 2 page 32)*
MULTISETS

- Don’t confuse \( \leq \oplus \) with multiset extension \( \leq_{\text{mul}} \): 
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- **Thm** (Dershowitz, Manna 1979): if \( < \text{wf} \) then \( <_{\text{mul}} \text{wf} \).
  *(Proof sketch: see footnote 2 page 32)*

- **Prop 1.22**: if \( \leq \) antisymmetric wqo then \( \leq_{\text{mul}} \) wqo.
  Proof: \( \leq_{\text{mul}} \) extends \( \leq \oplus \) (why?) then use Lemma 1.13 (2). \( \square \)
FINITE SETS

- Let $P_f(D) = \text{set of finite subsets of } D$

**Hoare quasi-ordering** (a.k.a., domination) $\leq^b$:

$A \leq^b B \iff \forall a \in A, \exists b \in B, a \leq b$

iff $\downarrow A \subseteq \downarrow B$ (why?)
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- Proof: $\sigma : D^\circ \to \mathcal{P}_f(D)$ is surjective and monotonic. □
Let $\mathcal{H}_{\text{fin}}(D)$ = set of downward closures $\downarrow A$ of finite subsets $A$ of $D$ ordered by $\subseteq$.
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**Corollary 1.26**: If $\leq$ wqo on $D$, then $\subseteq$ wqo on $\mathcal{H}_{\text{fin}}(D)$. 
FINITE HOARE POWERSET

- Let $H_{\text{fin}}(D) =$ set of downward closures $\downarrow A$ of finite subsets $A$ of $D$ ordered by $\subseteq$

- **Corl 1.26**: If $\leq$ wqo on $D$, then $\subseteq$ wqo on $H_{\text{fin}}(D)$.

- Does **not** extend to full Hoare powerdomain $H(D) =$ set of all downwards-closed subsets of $D$ with $\subseteq$.

Neither to full powerset $P(D)$ with $\leq^b$ (see next slides).
**Thm 1.28 (I):** $R$ is wqo.

Let $(a_n, b_n)_{n \in \mathbb{N}}$ in $R$. By Dickson (on $\mathbb{N}^2$, not $R$!) we may assume $(a_n, b_n)_{n \in \mathbb{N}}$ monotonic.
RADO’S STRUCTURE (1954)

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- If \( (a_n)_{n \in \mathbb{N}} \) stationary, done (why?)

\[
R \overset{\text{def}}{=} \{(a, b) \in \mathbb{N}^2 \mid a < b\}
\]

\[
(a, b) \leq (a', b') \overset{\text{def}}{\iff} \begin{cases} a = a' \land b \leq b' \\ b < a' \end{cases}
\]

\[
L_n \overset{\text{def}}{=} \{(a, n) \in R \mid a < n\}
\]

\[
C_n \overset{\text{def}}{=} \{(n, b) \in R \mid n < b\}
\]
RADO’S STRUCTURE (1954)

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- If \((a_n)_{n \in \mathbb{N}}\) stationary, done (why?)

- Else \( a_n \to \infty \), so \( b_0 < a_n \) for some \( n \).
  Then \((a_0, b_0) \leq (a_n, b_n)\). □
Thm 1.28 (1): $R$ is wqo.
(2) $\mathcal{P}(R), \mathcal{H}(R)$ are not.

Let $\omega_i = C_i \cup \{(a, b) \mid a < b \leq i - 1\}$: downwards-closed in $R$ (why?)
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- If $(\omega_i)_{i \in \mathbb{N}}$ were good, there would be $i < j$ such that $\omega_i \subseteq \omega_j$.

- But $(i, j) \in \omega_i, \not\in \omega_j$. \[\square\]
The following example illustrates the semantics, we establish bounds on the value, on which order constraints may be placed. We adopt order constants representing points in a linearly ordered domain. These atomic facts, together with facts of the forms $d < p$, may be thought of as a special sort of null assertion order relations between certain constants $A, B$ and $J$. The security guard's log shows agent $A$ entering the compound and claims that while there, agent $B$ must have removed the document, copied it, and then replaced it. Thus the culprit was in the compound at least twice. The security guard's log shows agent $A$ entering the compound, then leaving. Some time later, $A$ also came into the compound and claims that while there, agent $B$ also came into the compound. Agent $A$ says he left the night, so this is all the information his log shows. He is dishonorably discharged for dereliction of duty. Interrogation of agent $A$ and agent $B$ yields the following information: agent $A$ admits to having been in the compound twice. The security guard's log may be expressed as $\exists x \forall y (x < y \land y < x)$. In applications dealing with ordered domains, the available data is frequently indefinite. While the domain is actually linearly ordered, only some of the order relations holding between points in the data are known. Thus, the data provides only a partial order, and query answering involving determining what holds under all the compatible linear orderings is intractable even under the data complexity measure, but identify a number of PTIME subproblems. Data contains only a partial order, and query answering is equivalent to the problem of containment of conjunctive relational quasi-order techniques. We also show that the query problem we study is equivalent to the problem of containment of conjunctive relational databases containing such indefinite information. We show that one of our result implies that $\mathcal{L}_{<}$-complete, solving an open problem of Klug (1997).
INDEFINITE TEMPORAL DATABASES

- Indefinite temporal database $D = \text{finite set of}$
  - \textbf{ground atoms} $P(c_1,\ldots,c_n)$ (rows of a relation)
  - \textbf{temporal atoms} $c \leq d, c < d$ (ordering instants)
  
  where $c, d, \ldots$ are constants

- We will consider \textbf{monadic} ground atoms only, viz. $n = 1$,
  
  i.e. all ground atoms are of the form $P(c)$
  
  and ignore non-temporal constants (easy to deal with separately).

- \textbf{Models} map each constant $c$ to a value in $\mathbb{Z}$,
  
  each predicate $P$ to a subset of $\mathbb{Z}$
  
  so that all atoms of $D$ are satisfied.
DISJUNCTIVE MONADIC QUERIES

- $\varphi ::= P(t) \mid s \leq t \mid s < t \mid \exists x, \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi$

- $t ::= x \mid c$

- $D \models \varphi$ iff all models of $D$ satisfy $\varphi$. 
DISJUNCTIVE MONADIC QUERIES

- \( \varphi ::= P(t) | s \leq t | s < t | \exists x, \varphi | \varphi \land \varphi | \varphi \lor \varphi \)
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- **Thm** (van der Meyden 1997): The following is coNP-complete:
  - INPUT: \( D, \varphi \)
  - QUESTION: \( D \models \varphi \)?
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- **Thm** (van der Meyden 1997): The following is $\text{coNP}$-complete:
  - INPUT: $D, \varphi$
  - QUESTION: $D \models \varphi$?

- **Thm** (ibidem): For fixed $\varphi$, the following is solvable in **linear time**:
  - INPUT: $D$
  - QUESTION: $D \models \varphi$?
- Databases $D$ are assumed \textbf{normalized}:
  no cycle $c_1 \sqsubset \ldots \sqsubset c_n = c_1$
  (where each $\sqsubset$ is $\leq$ or $<\$).
Databases $D$ are assumed **normalized**: no cycle $c_1 \sqsubseteq \ldots \sqsubseteq c_n = c_1$ (where each $\sqsubseteq$ is $\leq$ or $<)$.

Normalization in **linear time** by Tarjan’s strong connected component algorithm:
— non-trivial SCCs with at least one $<$ edge $\Rightarrow$ inconsistent
— else replace all vertices $c_i$ of an SCC by the same constant
Every (normalized) database can be represented as a DAG

\[ D = \{P, Q\} < \{P\} < \{S\} \leq \{R\} \]
Every (normalized) database can be represented as a DAG

Each path encoded as a **flexiword** = word on \( \Sigma = \mathcal{P}(Predicates) \cup \{\leq, ' '<\} \)

\[
\begin{align*}
D &= \\
&= \{P(c_1), Q(c_1), P(c_2), R(c_3), S(c_4), c_1 < c_2, c_2 < c_4, c_2 \leq c_3\}
\end{align*}
\]
FLEXIWORDS

- Every (normalized) database can be represented as a DAG
- Each path encoded as a flexiword = word on $\sum = \mathcal{P}(Predicates) \cup \{\leq, '<'\}$
- Here:  
  - $\{P,Q\} '<' \{P\} '<' \{S\}$
  - $\{P,Q\} '<' \{P\} '\leq' \{R\}$

Diagram:
- $D = \{P(c_1), Q(c_1), P(c_2), R(c_3), S(c_4), c_1 < c_2, c_2 < c_4, c_2 \leq c_3\}$
- Edge $C_1 \leq C_2$
- $\{P\} < C_4$
- $\{R\} \leq C_3$
A WQO ON DATABASES

- **Flexiwords** = words on $\Sigma = P(Predicates) \cup \{\leq, \prec\}$
- Databases $D \cong$ sets of flexiwords, in $P_f(\Sigma^*)$
A WQO ON DATABASES

- **Flexiwords** = words on $\Sigma = \mathcal{P}(\text{Predicates}) \cup \{'\leq', '<'\}$

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- Order letters by $\sqsubseteq$ the disjoint union of:
  - $\subseteq$ on $\mathcal{P}(\text{Predicates})$
  - $'\leq'$ below $'<'$
A WQO ON DATABASES

- **Flexiwords** = words on $\Sigma = \mathcal{P}(Predicates) \cup \{\leq, '<'\}$
  
  Databases $D \cong$ sets of flexiwords, in $\mathcal{P}_f(\Sigma^*)$

- Order letters by $\sqsubseteq$ the disjoint union of:
  
  — $\subseteq$ on $\mathcal{P}(Predicates)$
  
  — '$\leq$' below '$<$'

- **Note:** $(\sqsubseteq^*)^b$ is wqo on $\mathcal{P}_f(\Sigma^*)$. 
THE KEY LEMMA

- **Lemma**: If $D \models \varphi$ and $D \left( \subseteq^* \right) \vdash D'$, then $D' \models \varphi$ (admitted).
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- **Lemma:** If $D \models \varphi$ and $D \ (\subseteq *) \vdash D'$, then $D' \models \varphi$ (admitted).

- Hence $A_{\varphi} = \{D \mid D \models \varphi\}$ is upwards-closed.
**THE KEY LEMMA**

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- Hence $A_\varphi = \{D \mid D \models \varphi\}$ is upwards-closed.

- Hence $A_\varphi = \uparrow A_0$ for some finite subset $A_0$ of $P_f(\Sigma^*)$. 
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- Hence $A_\varphi = \{D \mid D \models \varphi\}$ is upwards-closed.

- Hence $A_\varphi = \uparrow A_0$ for some finite subset $A_0$ of $\mathcal{P}_f(\Sigma^*)$.

- **Note:** for fixed $D$, checking $D \ (\sqsubseteq^\ast) \ D'$ can be done in linear time in $|D'|$, using memoization.
VAN DER MEYDEN’S ALGORITHM

- Let $A_0$ be a finite basis of $A_\varphi$. 
VAN DER MEYDEN’S ALGORITHM

- Let $A_0$ be a finite basis of $A_\varphi$.

- Linear time algorithm:
  INPUT: $D$
  for each $D' \in A_0$ do:
  | if $D' (\sqsubseteq^*) \vdash D$ then accept; ($* D \models \varphi *$)
  reject; □
THE MAGIC

- There is a linear time algorithm… but we cannot implement it! (why?)
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  but we cannot implement it! (why?)

- Solved by Ogawa (2003):
  — Higman’s Lemma has a **constructive proof**
  — by the Curry-Howard correspondence,
    this yields an algorithm to compute $A_0$ from $\varphi$
    (details omitted)
  — complexity: unknown, probably pretty high. ☐
PARAMETERIZED COMPLEXITY

- **Corollary**: deciding disjunctive queries on monadic indefinite temporal databases is in **FPT**.

- **FPT** (« fixed parameter tractable ») = class of pbs of the form:
  
  **PARAMETER**: $y$  
  **INPUT**: $x$  
  **QUESTION**: $(x,y) \in L$?  
  
  solvable in $\text{TIME}(f(|y|) \cdot \text{poly}(|x|))$ where $f$ is computable.
**PARAMETERIZED COMPLEXITY**

- **Corl:** deciding disjunctive queries on monadic indefinite temporal databases is in **FPT**.

- **FPT** (« fixed parameter tractable ») = class of pbs of the form:
  
  **PARAMETER:** $y$ (typically, assumed small, such as $\varphi$)  
  **INPUT:** $x$ (e.g., $D$)  
  **QUESTION:** $(x,y) \in L$? (e.g., $D \models \varphi$?)  
  
  solvable in $\text{TIME}(f(|y|).\text{poly}(|x|))$ where $f$ is computable.

- **Note:** $\text{poly}(|x|)$ **independent** of $|y|$.
**Kruskal’s Theorem**

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TREES = 1ST-ORDER TERMS

- Let $D$ be a signature = set of ‘function symbols’ … not necessarily finite

- Terms over $D$:
  $$s, t, \ldots ::= f(s_1, \ldots, s_m) \quad f \in D$$

- Note: base case hidden
  (take $m=0$, then $a()$ written $a$)

- Same as finite trees with $D$-labeled vertices
TREE EMBEDDING
(a.k.a. homeomorphic embedding)

\[ f(a, b) \leq_T f(g(a, h(b, c)), b) \]

To go up in \( \leq_T \):
- increase symbols
- insert symbols vertically
- add whole subterms as sons of existing nodes

assuming
\[ f \leq f \]
\[ a \leq a \]
TRE EMBEDDING

- Set $\mathcal{T}'(D)$ of terms over $D$:
  
  $s, t, \ldots ::= f(s_1, \ldots, s_m) \quad f \in D$
  
  i.e., $f(s)$ where $f \in D, s \in \mathcal{T}'(D)^*$

- Note: $\mathcal{T}'(D) = D \times \mathcal{T}'(D)^*$
**TREE EMBEDDING**

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  s, t, \ldots ::= f(s_1, \ldots, s_m) \quad f \in D
  \]
  
i.e., $f(s)$ where $f \in D, s \in \mathcal{T}(D)^*$

- Note: $\mathcal{T}(D) = D \times \mathcal{T}(D)^*$

- Formally, $\leq_T$ defined by the rules:
  
  $\begin{align*}
  s \leq_T t_i & \quad \text{(T-add)} \\
  s \leq_T f(t_1 \cdots t_n) & \quad \text{(T-inc)} \\
  f \leq g & \quad s (\leq_T)^* t \\
  f(s) \leq_T g(t)
  \end{align*}$
KRUSKAL’S THEOREM (1960)

- **Thm 1.29**: If \( \leq \) is wqo on \( D \), then \( \leq_T \) is wqo on \( \mathcal{T}(D) \).

- Proof by minimal bad sequence argument, as with Higman, but:
  - will require Higman as an auxiliary lemma
  - one additional (clever) trick.
**KRUSKAL’S THEOREM (1960)**

- **Thm 1.29**: If $\leq$ is wqo on $D$, then $\leq_T$ is wqo on $\mathcal{T}(D)$.

- Proof by minimal bad sequence argument, as with Higman, but:
  - will require Higman as an auxiliary lemma
  - one additional (clever) trick.
MINIMAL BAD SEQUENCES

• **Subterm ordering**: \( s \triangleleft t \) iff \( s \) is a subterm of \( t \), i.e.,
  \( s = t \) or \( (t = f(t_1, \ldots, t_n) \) and \( s \triangleleft t_i \) for some \( i, 1 \leq i \leq n \), inductively
  **Note**: this is wf.

• **Bad sequence**: \((t_n)_{n \in \mathbb{N}}\) such that no \( t_m \leq_T t_n \) for any \( m < n \).
  Let **Bad** be the set of bad sequences.

• \((t_n)_{n \in \mathbb{N}} (\triangleleft)_{\text{lex}} (t'_n)_{n \in \mathbb{N}}\) iff [same sequence or] for some \( n \),
  \[ t_0 = t'_0, \quad t_1 = t'_1, \quad \ldots, \quad t_{n-1} = t'_{n-1}, \quad t_n \triangleleft t'_n. \]
MINIMAL BAD SEQUENCES

• **Subterm ordering:** \( s \trianglelefteq t \) iff \( s \) is a subterm of \( t \), i.e., \( s = t \) or \( t = f(t_1, \ldots, t_n) \) and \( s \trianglelefteq t_i \) for some \( i, 1 \leq i \leq n \), inductively

  **Note:** this is wf.

• **Bad sequence:** \((t_n)_{n \in \mathbb{N}}\) such that no \( t_m \leq_T t_n \) for any \( m < n \).

  Let \textbf{Bad} be the set of bad sequences.

• \((t_n)_{n \in \mathbb{N}} \leq_{\text{lex}} (t'_n)_{n \in \mathbb{N}}\) iff [same sequence or] for some \( n \),

  \( t_0 = t'_0, t_1 = t'_1, \ldots, t_{n-1} = t'_{n-1}, t_n \triangleleft t' \).
MINIMAL BAD SEQUENCES

- **Subterm ordering**: \( s \preceq t \) iff \( s \) is a subterm of \( t \), i.e., \( s = t \) or \( t = f(t_1, \ldots, t_n) \) and \( s \preceq t_i \) for some \( i, 1 \leq i \leq n \), inductively.  
  **Note**: this is wf.

- **Bad sequence**: \((t_n)_{n \in \mathbb{N}}\) such that no \( t_m \leq_T t_n \) for any \( m < n \). Let \( \text{Bad} \) be the set of bad sequences.

- \((t_n)_{n \in \mathbb{N}} \preceq_{\text{lex}} (t'_n)_{n \in \mathbb{N}}\) iff [same sequence or] for some \( n \), \( t_0 = t'_0, t_1 = t'_1, \ldots, t_{n-1} = t'_{n-1}, t_n \prec t'_n \).

- As with Higman, if \( \text{Bad} \neq \emptyset \) then there is a minimal bad sequence.
KRUSKAL: THE PROOF (1/4)

- Let \((t_n)_{n \in \mathbb{N}}\) be \((\preceq)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n)\), \(f_n \in D\), \(s_n \in \mathcal{T}(D)^*\)
KRUSKAL: THE PROOF (1/4)

- Let \((t_n)_{n \in \mathbb{N}}\) be \((\preceq)\)\textsubscript{lex}-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in T(D)^*\)

- Since \(D\) wqo, \((f_n)_{n \in \mathbb{N}}\) is perfect. Find \(f_{i[0]} \leq f_{i[1]} \leq f_{i[2]} \leq \ldots\) with \(i[0] < i[1] < i[2] < \ldots\)
KRUSKAL: THE PROOF (1/4)

• Let \((t_n)_{n \in \mathbb{N}}\) be \(\langle \rangle_\text{lex}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in \mathcal{T}(D)^*\)

• Since \(D\) wqo, \((f_n)_{n \in \mathbb{N}}\) is perfect. Find \(f_i[0]<f_i[1]<f_i[2]<\ldots\) with \(i[0]<i[1]<i[2]<\ldots\)

• Oops, this is the point where we need the additional clever trick…
KRUSKAL: THE PROOF (2/4)

- Let \((t_n)_{n \in \mathbb{N}}\) be \((\preceq)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in T(D)^*\)… explicitly \(s_n = s_{n1}, \ldots, s_{nm[n]}\)
KRUSKAL: THE PROOF (2/4)

- Let \((t_n)_{n \in \mathbb{N}}\) be \((\triangleleft)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in \mathcal{T}(D)^*\) ... explicitly \(s_n = s_n 1, \ldots, s_n m[n]\)

- **Claim:** Let \(U = \{s_{nk} \mid n \in \mathbb{N}, 1 \leq k \leq m[n]\}\). Then \(\leq_T\) is wqo on \(U\).
  (This requires that \((t_n)_{n \in \mathbb{N}}\) be \textbf{minimal} bad!)
KRUSKAL: THE PROOF (2/4)

- Let \((t_n)_{n \in \mathbb{N}}\) be \((\triangleleft)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in T(D)^*\)... explicitly \(s_n = s_{n1},...,s_{nm[n]}\)

- **Claim:** Let \(U = \{s_{nk} | n \in \mathbb{N}, 1 \leq k \leq m[n]\}\). Then \(\leq_T\) is wqo on \(U\).
  (This requires that \((t_n)_{n \in \mathbb{N}}\) be \textbf{minimal} \textbf{bad}!)

- **Proof (1/2):** Imagine there is a bad sequence \((s_{n[i]} k[i])_{i \in \mathbb{N}}\) in \(U\).
  Note that \(n[i]\) is uniquely determined from \(s_{n[i]} k[i]\) (**why**?)
  Remove those finitely many (**why**) elements such that \(n[i] < n[0]\).
  Result is again bad (**why**)?
  Hence we can assume that \(n[i] \geq n[0]\) for every \(i\).
KRUSKAL: THE PROOF (3/4)

• Let \((t_n)_{n \in \mathbb{N}}\) be \((\preceq)_{\text{lex}}\)-minimal in \textbf{Bad}.

Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in \mathcal{T}(D)^\ast\ldots\) explicitly \(s_n = s_n1, \ldots, s_n m[n]\)

• \textbf{Claim}: Let \(U = \{s_{nk} \mid n \in \mathbb{N}, 1 \leq k \leq m[n]\}\). Then \(\leq_T\) is wqo on \(U\).

(This requires that \((t_n)_{n \in \mathbb{N}}\) be \textbf{minimal} \textbf{bad!})

• Proof (2/2): Imagine there is a bad sequence \((s_n[i] k[i])_{i \in \mathbb{N}}\) in \(U\) \((n[i] \geq n[0]\) for every \(i\)).

\(t_0, t_1, \ldots, t_{n[0]-1}, S\mathcal{n}[0] k[0], S\mathcal{n}[1] k[1], \ldots, S\mathcal{n}[i] k[i], \ldots\) is \textbf{good} (why?)

− case 1: \(t_m \leq_T t_n\) for some \(m< n\geq n[0]\): contradiction (why?)

− case 2: \(t_m \leq_T s_n[i] k[i]\) for some \(m< n\) and \(i \geq 0\): contradiction (why?)

− case 3: \(s_n[i] k[i] \leq_T s_n[j] k[j]\) for some \(i < j\): contradiction (why?) \(\square\)
KRUSKAL: THE PROOF (4/4)

• Let \((t_n)_{n \in \mathbb{N}}\) be \((\sqsubseteq)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in D, s_n \in \mathcal{T}(D)^*\)… explicitly \(s_n = s_{n1}, \ldots, s_{nm[n]}\)

• **Claim:** Let \(U = \{s_{nk} \mid n \in \mathbb{N}, 1 \leq k \leq m[n]\}\). Then \(\leq_T\) is wqo on \(U\). □
KRUSKAL: THE PROOF (4/4)

• Let \((t_n)_{n \in \mathbb{N}}\) be \((\trianglelefteq)_{\text{lex}}\)-minimal in \textbf{Bad}.
  Write \(t_n\) as \(f_n(s_n), f_n \in \mathcal{D}, s_n \in \mathcal{T}(\mathcal{D})^*\) … explicitly \(s_n = s_{n1}, \ldots, s_{nm[n]}\)

• \textbf{Claim}: Let \(U = \{s_{nk} \mid n \in \mathbb{N}, 1 \leq k \leq m[n]\}\). Then \(\leq_T\) is wqo on \(U\). \(\square\)

• By Dickson+Higman, \(\leq x (\leq_T)^*\) is also wqo on \(D \times U^*\).
KRUSKAL: THE PROOF (4/4)

• Let \( (t_n)_{n \in \mathbb{N}} \) be \((\preceq)_{\text{lex}}\)-minimal in \textbf{Bad}.

Write \( t_n \) as \( f_n(s_n), f_n \in D, s_n \in T(D)^* \)… explicitly \( s_n = s_{n1}, ..., s_{nm[n]} \)

• **Claim:** Let \( U = \{s_{nk} \mid n \in \mathbb{N}, 1 \leq k \leq m[n]\} \). Then \( \preceq_T \) is wqo on \( U \). \( \square \)

• By Dickson+Higman, \( \preceq x (\preceq_T)^* \) is also wqo on \( D \times U^* \).

• Hence for some \( m < n \), \( f_m \leq f_n \) and \( s_m (\preceq_T)^* s_n \): contradiction (\textbf{why?}) \( \square \)
APPLICATION: TERMINATION OF REWRITE SYSTEMS
REWRITE SYSTEMS

- Terms now given over a signature $\Sigma$ specifying **arities** for each function symbol
- Rewrite system $R$:
  set of **rules** $l \rightarrow r$
  where $l, r$ are terms applicable under every context
- **Pb:** show **termination**

| $D_x(x)$   | $\rightarrow$ | 1  |
| $D_x(a)$   | $\rightarrow$ | 0  |
| $D_x(M + N)$ | $\rightarrow$ | $D_x(M) + D_x(N)$ |
| $D_x(M \times N)$ | $\rightarrow$ | $D_x(M) \times N + M \times D_x(N)$ |
| $D_x(M - N)$ | $\rightarrow$ | $D_x(M) - D_x(N)$ |
| $D_x(-M)$   | $\rightarrow$ | $-D_x(M)$ |
| $D_x(M/N)$  | $\rightarrow$ | $\left( D_x(M) \times N - M \times D_x(N) \right) / N^2$ |
| $D_x(\log(M))$ | $\rightarrow$ | $D_x(M)/M$ |
| $D_x(M^N)$  | $\rightarrow$ | $N \times (M^{N-1} \times D_x(M)) + M^N \times (\log(M) \times D_x(N))$ |
MULTISET PATH ORDERING

- Fix a strict ordering $<$ on $\Sigma$

\[
(\exists i,) \quad s \leq_{\text{rpo}} t_i \\
\quad s <_{\text{rpo}} f(t_1, \ldots, t_n)
\]

\[
f < g \quad t = g(t_1, \ldots, t_n) \quad \forall i, s_i <_{\text{rpo}} t \\
f(s_1, \ldots, s_m) <_{\text{rpo}} t
\]

\[
f = g \quad t = g(t_1, \ldots, t_n) \quad \forall i, s_i <_{\text{rpo}} t \\
|s_1, \ldots, s_m| \leq_{\text{rpo mul}} |t_1, \ldots, t_n| \\
f(s_1, \ldots, s_m) <_{\text{rpo}} t
\]

What about taking $D_x > 1, 0, a, x, +, -, /, ^, \log$?

\[
\begin{align*}
D_x(x) & \to 1 \\
D_x(a) & \to 0 \\
D_x(M + N) & \to D_x(M) + D_x(N) \\
D_x(M \times N) & \to D_x(M) \times N + M \times D_x(N) \\
D_x(M - N) & \to D_x(M) - D_x(N) \\
D_x(-M) & \to -D_x(M) \\
D_x(M/N) & \to \left( D_x(M) \times N - M \times D_x(N) \right)/N^2 \\
D_x(\log(M)) & \to D_x(M)/M \\
D_x(M^N) & \to N \times (M^{N-1} \times D_x(M)) \\
& \quad + M^N \times (\log(M) \times D_x(N))
\end{align*}
\]
**MULTISET PATH ORDERING**

- **Thm** (Dershowitz 1982, see also JGL 2001):
  If $\prec$ is wf, then $\prec_{rpo}$ is wf.

- $\prec_{rpo}$ is:
  - monotonic
  - compatible with substitution
  - a **simplification ordering**: $s \triangleleft t$ implies $s \prec_{rpo} t$ *(why?)*

- Hence if $r \prec_{rpo} l$ for every rule $l \rightarrow r$ of $R$, then $R$ terminates
SIMPLIFICATION ORDERINGS

- Let \( \subseteq \) be a **monotone simplification** ordering. (\( s_i \subseteq t_i \) implies \( f(\ldots, s_i, \ldots) \subseteq f(\ldots, t_i, \ldots) \))
- Let \( \leq_T \) be tree embedding wrt. = on function symbols in \( \Sigma \)

- **Lemma 1.43**: \( s \leq_T t \Rightarrow s \subseteq t \) (why?)
SIMPLIFICATION ORDERINGS

- Let \( \sqsubseteq \) be a **monotone** simplification ordering. (\( s_i \sqsubseteq t_i \) implies \( f(\ldots, s_i, \ldots) \sqsubseteq f(\ldots, t_i, \ldots) \))

- **Lemma 1.43**: \( s \leq_T t \Rightarrow s \sqsubseteq t \) (why?)

- **Corollary** (Thm 1.44; Dershowitz 1982 again):
  If \( \Sigma \) **finite**, then **every** simplification ordering is **wf**.
SIMPLIFICATION ORDERINGS

- Let \( \subseteq \) be a **monotone** simplification ordering. (\( s_i \subseteq t_i \) implies \( f(\ldots, s_i, \ldots) \subseteq f(\ldots, t_i, \ldots) \))

- **Lemma 1.43**: \( s \leq_T t \Rightarrow s \subseteq t \) (why?)

- **Corollary** (Thm 1.44; Dershowitz 1982 again): If \( \Sigma \) **finite**, then every simplification ordering is wf.

- Proof: assume \( t_0 \not\subseteq t_1 \not\subseteq \ldots \not\subseteq t_m \not\subseteq \ldots \)

  By Kruskal (why is \( \Sigma \) finite needed here?),

  \( t_m \leq_T t_n \) for some \( m<n \). Contradiction (why?). \( \Box \)