Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

1 Job scheduling: a greedy variant

We consider \( n \) jobs \( J_1, J_2, \ldots, J_n \), with processing times \( p_1, p_2, \ldots, p_n \), respectively, that we have to be scheduled on \( m \) identical machines. We consider the following algorithm: at first, we determine the \( m \) jobs with the largest processing times and we schedule them one per machine. The remaining \( n - m \) jobs are assigned to the machines greedily, i.e. at every time that a machine becomes idle, an arbitrary unscheduled job is assigned on it. Our objective is the minimization of the makespan, i.e. the time at which the last task finishes its execution.

(a) What is the approximation ratio of the algorithm described above? (Give the analysis.)

(b) Give a worst case example for this algorithm.

2 Integrality gap

(a) We are given a set of items \( I = \{1, 2, \ldots, n\} \). Each item \( i \) has its value \( v_i \) and its size \( s_i \). We are given a value \( D \) and the objective is to choose a subset \( A \subseteq I \) of items of minimum total size such that the total value is at least \( D \).

Let \( x_i \) be a binary variable that indicates whether item \( i \) is chosen to be in the solution or not. The following integer linear program models the problem:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} s_i x_i \\
\text{s.t.} & \quad \sum_{i=1}^{n} v_i x_i \geq D \\
& \quad x_i \in \{0, 1\}, \quad \forall i \in I
\end{align*}
\]

What is the integrality gap of the corresponding linear programming relaxation where the integrality constraints are replaced by \( x_i \geq 0 \)?

(b) What is the integrality gap of the linear program that we introduced in class for the generalized load balancing problem?