Problem 1 Given a graph $G = (V, E)$, and a subset of edges $S \subseteq E$, let $c(S)$ denote the number of connected components of the subgraph $(V, S)$.

(a) For each of the following possibilities:

- monotone, submodular, supermodular, modular,

explain (and give a proof or a convincing argument) whether the function $c$ has the above property.

(b) Let $\bar{c}(S) = c(E \setminus S)$ be the number of connected components in $G$ when we remove $S$. Repeat the question of (a), this time for the function $\bar{c}$.

(c) Give an application in which minimizing $\bar{c}$ can be useful. Be precise here, and do not attempt if you do not have at least a reasonable candidate of an application.

Problem 2 Consider the ski rental problem we saw in class. Let integer $B$ denote the cost of buying skis, and let $l$ denote the number of skiing days. The online algorithm does not know $l$.

In this problem we will derive a primal-dual online algorithm for the ski rental problem. At first glance it is surprising that we can use linear programming to obtain an online algorithm: this is because in what we have seen so far, LP formulations typically assume that the entire input is known.

(a) Give an integer linear programming formulation for the offline problem, that is assuming $l$ is known to the algorithm. Use a variable $x_i$ concerning a decision to rent on day $i$ and a variable $z$ concerning the decision to buy or not. Explain in detail the entire LP: all constraints, all variables, and explain why the ILP is a formulation of the offline optimum problem.

(b) Give the LP relaxation of the ILP in (a) and its dual. Explain why it is a relaxation. Give an intuitive explanation of the constraints in the dual.

(c) We are now getting ready to obtain a primal-dual algorithm for the online problem. You should note that every new day introduces a new variable, a new constraint, and a modification of the objective function in the primal. What can you say about the dual LP? That is, how is the dual LP “updated” on every new day?

(d) Give a simple primal-dual algorithm for the online problem. You will have to update some dual variables and make some corresponding decisions for the dual. Explain very clearly all the actions of the algorithm.
(e) Show that the algorithm of step (d) has competitive ratio 2. Is there any relation between this algorithm and the one we saw in class?

Warning: Do NOT attempt (e) if you know that the algorithm you got is not 2-competitive. You will not get any marks.