Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

Problem 1. Lagrangian Dual

Consider the following LP and let us denote by $z^*$ its optimal value.

$$
\begin{align*}
\text{max} & \quad c^T x \\
\text{subj to} & \quad Ax \leq b \\
& \quad x \geq 0.
\end{align*}
$$

(a) Obtain the Lagrangian relaxation $L(\lambda)$ of the LP, where $\lambda$ is the vector of Lagrangian multipliers, as well as the dual of the LP.

(b) In the Lagrangian relaxation, what is the relation between $A^T \lambda$ and $c$ that must be satisfied in order to obtain some meaningful upper bound on $z^*$ using $L(\lambda)$? Justify.

(c) Using the previous question, what can you say about the “best” set of Lagrangian multipliers that gives the best upper bound on $z^*$?

Problem 2. Prophet Inequality

Consider the LP relaxation for the prophet inequality problem we saw in class.

(a) Let $w$ be such that

$$
\sum_j \Pr[X_j \geq w] = 1.
$$

Explain why such $w$ exists.

(b) Consider the following strategy for the gambler, that uses the $w$ defined in part (a): Choose an arbitrary order for the boxes, and go over them in that order. If box $j$ is encountered, flip a random coin, and do the following:

- skip the box with probability $1/2$.
- open the box with probability $1/2$ and observe $X_j$: If $X_j \geq w$, then choose box $j$ and stop. Otherwise, continue.

Prove that the above strategy encounters each box in the ordering with probability at least $1/2$.

(c) Give the approximation ratio of the strategy in (b). Carefully justify your answer.