Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

Consider the following algorithm for the Adwords problem, based on the notation we used in the lecture. We refer to vertices in the L bipartition as “buyers” and in the R bipartition as “items”.

- Initially, for each buyer $i$, $\alpha_i \leftarrow 0$.
- Upon arrival of new item $j$, allocate it to buyer $i$ maximizing $b_{ij}(1 - \alpha_i)$.
- If $\alpha_i \geq 1$, do nothing. Otherwise:
  1. Charge the buyer the minimum between $b_{ij}$ and his remaining budget and set $x_{ij} \leftarrow 1$.
  2. $\beta_j \leftarrow b_{ij}(1 - \alpha_i)$.
  3. $\alpha_i \leftarrow \alpha_i(1 + b_{ij}/B_i) + b_{ij}/((c - 1)B_i)$ ($c$ is determined below).

Define $R_{\text{max}} = \max_{i \in L, j \in R} \{b_{ij}/B_i\}$, and let $c = (1 + R_{\text{max}})^{1/R_{\text{max}}}$. Let $P$ and $D$ denote the primal and the dual solutions during the execution of the algorithm. Show the following.

(a) The algorithm produces a primal feasible solution.
(b) In each iteration, the changes in the primal and dual solutions are such that $\Delta P \leq (1 + (1/(c - 1))\Delta D$.
(c) Show by induction in $i$ that
$$\alpha_i \geq \frac{1}{c - 1}(\sum_{j \in R} b_{ij}x_{ij}B_i - 1).$$
(d) What is the competitive ratio of the algorithm? What do you get for $R_{\text{max}} \to 0$?

You may use the helpful inequality $\ln(1 + x)/x \geq \ln(1 + y)/y$, for all $0 \leq x \leq y \leq 1$. 