Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

Consider the following variant of the online secretary problem. All the assumptions are as in the standard version we discussed in class, with the exception that the algorithm can now select up to $k$ candidates, for a given, known integer $k$. In particular, in step $i$, the online algorithm can either select candidate $i$, or reject candidate $i$: these decisions are irrevocable. Moreover, at each step, the set of selected candidates cannot have cardinality greater than $k$. The value of the set of selected candidates is equal to the sum of the weights of all the candidates in the set. We say that an algorithm for this problem is $\alpha$-competitive if

$$E[v(S)] \geq \alpha E[v(S^*)],$$

where $S$ is the set of candidates selected by the algorithm, and $S^*$ is the optimal set of candidates, i.e., the set that maximizes the total value, subject to the constraint that $|S|$ and $|S^*|$ are at most $k$.

Consider the following class of algorithms (which extends the algorithm we saw in class). For some $t$ to be decided, the algorithm uses the first $t$ steps to build a reference set $R$, consisting of the $k$ candidates with the largest values seen during the first $t$ steps. These elements are kept for comparison, but not selected. Subsequently, when a candidate $i$ with value $v(i)$ is observed, a decision of whether to select $i$ into the set $S$ is made based on $v(i)$ and $R$, and the set $R$ is possibly updated. At any given time, let $j_1, j_2, \ldots, j_{|R|}$ be the elements of $R$, sorted by decreasing $v(j_i)$.

(a) Give an algorithm for this problem that belongs in the above class. The algorithm should have competitive ratio at most $1/e$. Be very clear in the description of the algorithm.

(b) Let $v_1^*, v_2^*, \ldots, v_k^*$ denote the $k$ largest elements of the set $\{v(1), v(2), \ldots, v(n)\}$, and for $a \in [1,k]$ let $i_a^* = v^{-1}(v_a^*)$ be the index in the sequence $v(i)$ at which $v_a^*$ appeared. For every such $a$, give a lower bound that your algorithm selects the candidate with value $v_a^*$, as function of $t$ and $n$.

(c) (This question is independent of the above). Show that

$$\sum_{i=t+1}^{n} \frac{1}{i-1} > \int_{t}^{n} \frac{dx}{x}.$$

(d) Combining (b) and (c) give the competitive guarantee of the algorithm.

Be careful: there are more than one natural algorithms for this problem, but one is much simpler to analyze than the others.