Asymptotic Consensus and Averaging Algorithms

Exercise 1. Let $\mathcal{S}_n$ be the set of stochastic matrices of size $n$, and let $\mathcal{M} \subseteq \mathcal{S}_n$ be a non-empty and finite subset of $\mathcal{S}_n$ such that any finite product of matrices in $\mathcal{M}$ is ergodic. We define the equivalence relation $\sim$ in $\mathcal{S}_n$ by:

$$A \sim B \iff G(A) = G(B),$$

where $G(A)$ and $G(B)$ are the graphs associated to $A$ and $B$, respectively.

1. Show that the relation $\sim$ is preserved by right (or left) multiplication, i.e.,

$$\forall A, B, M \in \mathcal{S}_n, \quad A \sim B \Rightarrow AM \sim BM.$$

2. Suppose that $A$ and $B$ are two equivalent matrices, i.e., $A \sim B$. Prove that $N(A) = 1$ if and only if $N(B) = 1$.

Let $A_0, A_1, \ldots, A_{n^2}$ a sequence of $n^2 + 1$ matrices in $\mathcal{M}$.

3. Show that there exist two indices $k$ and $\ell$, $0 \leq k < \ell \leq n^2$, such that $A_{n^2} \cdots A_k \sim A_{n^2} \cdots A_\ell$.

4. Prove that $A_{n^2} \cdots A_0$ is a scrambling matrix, i.e., $N(A_{n^2} \cdots A_0) < 1$.

5. What extension of Corollary 6 (cf. the course notes) have you just proved?

Exercise 2. A stochastic matrix $A$ is said to be doubly stochastic if its transpose $A^T$ is also stochastic.

1. Define a class of stochastic matrices that are all doubly stochastic.

2. What is the Perron vector of a doubly stochastic matrix?

Exercise 3. Let $G = ([n], E)$ be a symmetric and connected graph, and let $A$ be the stochastic matrix such that

$$A_{i,j} = 1/d_i,$$

where $d_i = d_i^- = d_i^+$ is the in-degree (or outdegree) of the node $i$ in $G$.

1. Show that the $i$-th entry of the Perron vector of $A$ is equal to $\pi_i = d_i/|E|$.

Let $q_1, \ldots, q_n$ be $n$ integers such that $q_i \geq d_i$, and let $B$ the $n \times n$ matrix defined by

$$A_{i,j} = 1/q_i.$$

2. Verify that $B$ is a stochastic matrix.

3. What is the Perron vector of $B$? What property do the FixedWeight and the Metropolis algorithms share?
Exercise 4. Let us consider the \textit{m-butterfly graph} depicted in Figure 1: It has \( n = 2m \) nodes and consists of two isomorphic parts that are connected by a bidirectional edge. We list the edges between the nodes 1, 2, \ldots, \( m \), which also determine the edges between the nodes \( m+1, m+2, \ldots, 2m \) via the isomorphism \( \bar{p} = 2m - p + 1 \). The edges between the nodes 1, 2, \ldots, \( m \) are: (a) the edges \((p+1,p)\) for all \( p \in [m-1] \) and (b) the edges \((1,p)\) for all \( p \in [m] \). In addition, it contains a self-loop at each node and the two edges \((m,\bar{m})\) and \((\bar{m},m)\). Hence the \( m \)-butterfly graph is strongly connected.

Let \( A \) be the stochastic matrix such that
\[
A_{i,j} = 1/d_i^+,
\]
where \( d_i^- \) is the in-degree of the node \( i \) in the Butterfly graph.

1. Verify that \( A \) is an ergodic matrix and its Perron vector is given by
\[
\pi_1 = \frac{1}{5}, \quad \pi_p = \frac{3}{5 \cdot 2^p} \text{ for } p \in \{2, \ldots, m-1\} \quad \text{and} \quad \pi_m = \frac{3}{5 \cdot 2^{m-1}}.
\]

2. Compare this Perron Vector with the one in Exercise 3, question 1.

Exercise 5. An averaging algorithm is said to be \( \alpha \)-\textit{safe} for a dynamic network \( G \) if, in every execution of this algorithm with the communication network \( G \), all positive weights are at least equal to \( \alpha \).

1. We consider an \( \alpha \)-safe averaging algorithm in a dynamic network \( G \), and an execution of this algorithm with \( G \). Prove that at every round \( t \) and for every agent \( i \), the output variable \( x_i \) satisfies
\[
(1 - \alpha)m_i(t-1) + \alpha M_i(t-1) \leq x_i(t) \leq (1 - \alpha)M_i(t-1) + \alpha m_i(t-1),
\]
where \( m_i(t-1) = \min_{j \in I_{n_i}(t)} x_j(t-1) \), \( M_i(t-1) = \max_{j \in I_{n_i}(t)} x_j(t-1) \), and \( I_{n_i}(t) \) denotes the set of \( i \)'s incoming neighbors in \( G(t) \).

2. Does the following inequalities:
\[
(1 - \alpha)m(t-1) + \alpha M(t-1) \leq x_i(t) \leq (1 - \alpha)M(t-1) + \alpha m(t-1),
\]
where \( m(t-1) = \min_{j \in [n]} x_j(t-1) \), \( M(t-1) = \max_{j \in [n]} x_j(t-1) \), hold?
Let $G = ([n], E)$ be a symmetric and connected graph.

3. Is the EqualNeighbor algorithm $\alpha$-safe in $G$? For what real number $\alpha$?

4. Same questions for the FixedWeight algorithm and the Metropolis algorithm.