MPRI 2-24-1: Algorithms and Uncertainty (2024)

Homework 4 — *Due on October 17, beginning of class*

Instructions You can write your solutions either in English or French. Please observe the homework policy as described in the course web page.

Consider the unweighted Set Cover problem with m sets and n elements, in which all sets have unit cost, as we discussed in class. Recall the distinction between the *fractional* and *integral* variants, as well as the *online* and *offline* variants. In class we gave an online algorithm for the fractional problem with competitive ratio $O(\log m)$.

(a) Show how to use randomized rounding for the offline problem, and obtain an offline algorithm for the integral version that has *approximation ratio* at most $O(\log n)$ (i.e., expected cost at most $O(\log n)$ times the optimum offline cost) and outputs a valid cover with at least a constant probability. Hint: Use the LP relaxation we discussed in class, and the AM-GM inequality.

(b) Consider now the online integral problem. Recall the algorithm discussed in the class for the fractional problem. Let x_S denote the primal variables and let Δx_S denote the change in x_S in each iteration (i.e., when an element e arrives). Last, let x_S^{-e} and x_S^{+e} be the respective values of x_S, before and after an element e arrives, hence $\Delta x_S = x_S^{+e} - x_S^{-e}$.

Consider the following online rounding scheme. Suppose that e arrives, and for a set S, x_S increases by Δx_s. Then:

1. If S is already in the solution then we do nothing.
2. Otherwise, we pick S with probability $\frac{\Delta x_S \log n}{1 - x_s \log n}$.

Let P_S^{-e} and P_S^{+e} denote the probability that S is picked before and after e arrives, respectively. Show that if $P_S^{-e} = x_S \log n$, then $P_S^{+e} = (x_s + \Delta x_s) \log n$.

Hint: The answer is no more than 3 lines. But be precise in your explanation, since we deal with probabilities.

(c) Explain how to combine the approaches of parts (a) and (b) and obtain an online algorithm for the integral version of competitive ratio $O(\log n \log m)$.